Phase-Space Networks of Frustrated Spin Models

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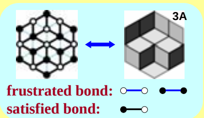
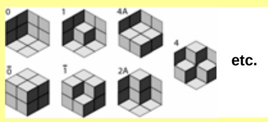
We propose a complex-network approach to study phase-space structures of two frustrated spin models. Their highly degenerated ground states are mapped as discrete networks such that the quantitative network analysis can be applied to phase-space studies for the first time. The resulting phase spaces share some common features and establish a class of complex networks with unique Gaussian spectral densities. A one-to-one correspondence is discovered between the six-vertex model (jigsaw puzzle) and sphere stack.

Models:

1. Antiferromagnetic Ising Spins on Triangular Lattice (ground state)

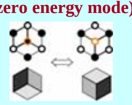
1-1 mapping to cube stacks





20 stacks ⇔ degeneracy = 20 ⇔

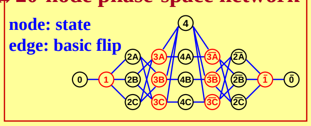
basic flip: (zero energy mode)



defines the edge

20-node phase-space network

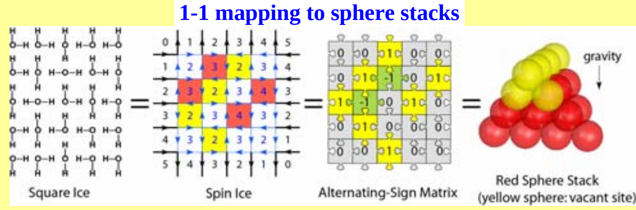
node: state
edge: basic flip



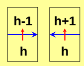
Dynamic processes at the microscopic level are time reversible ⇒ edges are undirected

2. six-vertex model (i.e. square ice, spin ice or jigsaw puzzle)

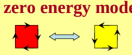
1-1 mapping to sphere stacks



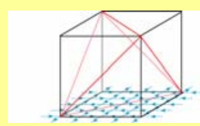
Height rule:
walking across a spin,
right arrow: -1; left arrow: +1.



Basic flip:
zero energy mode

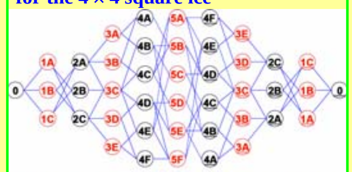


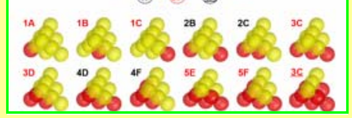
Corresponding to adding/removing a sphere



Domain-wall boundary condition ⇔ tetrahedron container

42-node phase-space network for the 4 × 4 square ice





Phase spaces are ergodic
except for periodic boundary conditions

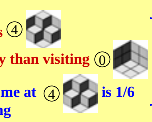
Fixed Boundary Conditions (can be viewed as stacks)

Mean visiting frequency of node i ∝ its connectivity k_i; staying time ∝ $1/k_i$

Example:

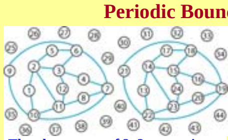
A random walker visits ④ 6 times more frequently than visiting ⓪

However, the staying time at ④ is 1/6 for random spin flipping



The staying time at each state is the same ⇒ ergodic

Periodic Boundary Conditions



The phase space of 2×3 square ice under periodic boundary condition

For m × n square ice, we proved that their phase-space networks have

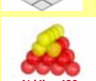
- $2^{m+1}+2^{n+1}-4$ **single nodes**
- $(m-1)\times(n-1)$ **nontrivial networks**
- $\frac{(m+n-1)!}{(m-1)!(n-1)!}$ **nodes in the smallest nontrivial networks**

Open questions in math

Cube stack is equivalent to the plane partition problem in combinatorics and number theory which has been intensively studied. However, the combinatoric properties of sphere stacking has not been explored. For example:

Number of ways to pack cubes in a box with side length *L*
***N*(*L*=1,2,3,4…) = 2, 20, 980, 232848… are give by the MacMahon formula:**

$$N_L=\prod_{i=1}^{L}\prod_{j=1}^{L}\prod_{k=1}^{L}\frac{i+j+k-1}{i+j+k-2}=\frac{H^3(L)H(3L)}{H^3(2L)}\sim\left(\frac{27}{16}\right)^{\frac{3}{2}L^2}\text{ when } L\to\infty,\quad H(L)=\prod_{k=1}^{L-1}k!$$



N (4) = 429

***N*(*L*=1,2,3,4…) = 2, 7, 42, 429, 7436, 218348…**
i.e. the alternating-sign matrix theorem

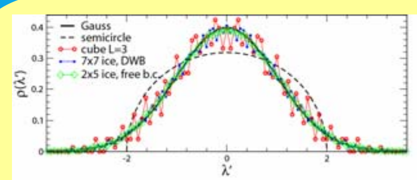
$$N_L=\prod_{1\le i\le j\le L+1}\frac{L+i+j}{2i+j-1}=\prod_{j=0}^{L}\frac{(3j+1)!}{(L+j+1)!}\sim\left(\frac{27}{16}\right)^{L^2/2}\text{ when } L\to\infty$$

N (4) = 868

Our numerical result: *N*(*L*=1,2,3,4…) = 2, 18, 868, 230274...

Formula for *N*(*L*) = ?

Gaussian spectral density
for all phase-space networks



Spectral Density ρ(λ):
the distribution of eigenvalues of the adjacency matrix
ρ(λ) characterize the topology of the network

- Phase-space networks: **Gaussian**
- Random networks: **semicircular (Wigner's semicircle law)**
- Scale free networks: **triangular with a power-law tail**
- Other networks: **irregular**

⇒ **Phase-space networks belong to a new class of networks with unique topology.**

Proof at the infinite-size limit:

$\rho(\lambda)$'s *q*th moment, M_q, is directly related to the network's topology.

$D_q=N_{\text{node}}M_q=\sum_{i=1}^{N_{\text{node}}}\lambda_i^q$ **is the number of paths (or loops) that return back to the original node after *q* steps.**

By counting loops in the network with the help of the cube/sphere-stack pictures, we can prove

$\boxed{D_{2n+1}=0\,,\ D_{2n}\cong(2n-1)!!\,\sigma^{2n}}$ ⟷ **Same as the moments of a Gaussian distribution ⇒ $\rho(\lambda)$ is Gaussian**

D_{2n} becomes exact at infinite-size limit. The variance

$$\sigma^2=D_2=\sum_i k_i=N_{\text{node}}\bar{k},\quad \bar{k}=\text{mean connectivity}$$

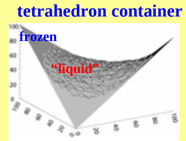
References:

Yilong Han, **Phys. Rev. E** 80, 051102 (**2009**)

Yilong Han, **Phys. Rev. E** 81, 041118 (**2010**)

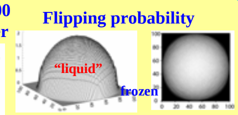
Boundary effects
directly visualized in 3D stacks

Sphere stack in L=100 tetrahedron container



Open question: what is the limit surface at *L* → ∞ ?

Flipping probability

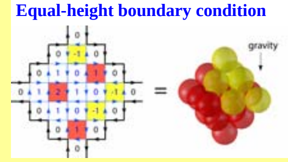


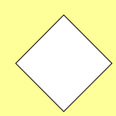
The "non-frozen" area is a circle – Arctic circle theorem
The local curvature of the limit shape is proportional to the density of free spin.

Boundary effects often percolate through the entire system ⇒ no thermodynamic limit

Different boundary conditions corresponds to different container shape

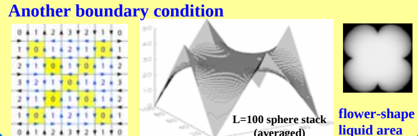
Equal-height boundary condition





all liquid, no frozen area

Another boundary condition



flower-shape liquid area

Common features

- **Small-world property**
- **Gaussian degree distribution**
- **Gaussian spectral density**
- **Fractal**