Brian Muenzenmeyer

Quantum state engineering using dissipation — dissipation-induced squeezing

GW & Harri Mäkelä, arXiv:1101.4845.

Gentaro Watanabe
渡辺　元太郎
(APCTP, RIKEN)

Contents

Charm of ultracold atom gases

Unique features of cold atom gases (1)

- Controllability & Flexibility

Manipulate system parameters

External pot.
Density ← Gas phase: low bulk energy
Strength & sign of the interaction ← Feshbach res.
Dimensionality
etc.

Statically & dynamically

⇒
- Understand quantum phenomena
- Quantum state engineering

Unique features of cold atom gases (2)

- <u>Observability & Measurability</u>

Microscopic scales are large

→ "Seeing is believing"

Size of ground st. of trap $d = \sqrt{\hbar/m\omega} = O(1)\mu\mathrm{m}$

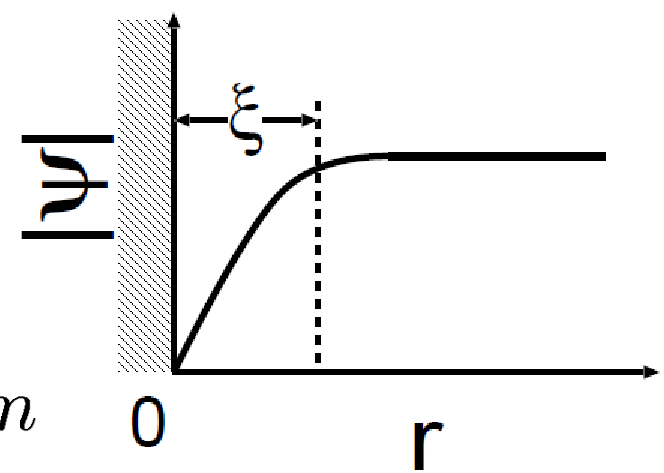
Optical lattice lattice const. ~ λ of lasers (sub-μm)

Healing length ξ ~ 1μm

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V + gn\right]\psi = \mu\psi$$

$$\Rightarrow \frac{\hbar^2}{m\xi^2} \sim gn$$

Effective int.
$g = 4\pi\hbar^2 a_s/m$



$$\xi \sim (r_s/a_s)^{1/2} r_s \sim 1\mu\mathrm{m}$$

Unique features of cold atom gases (3)

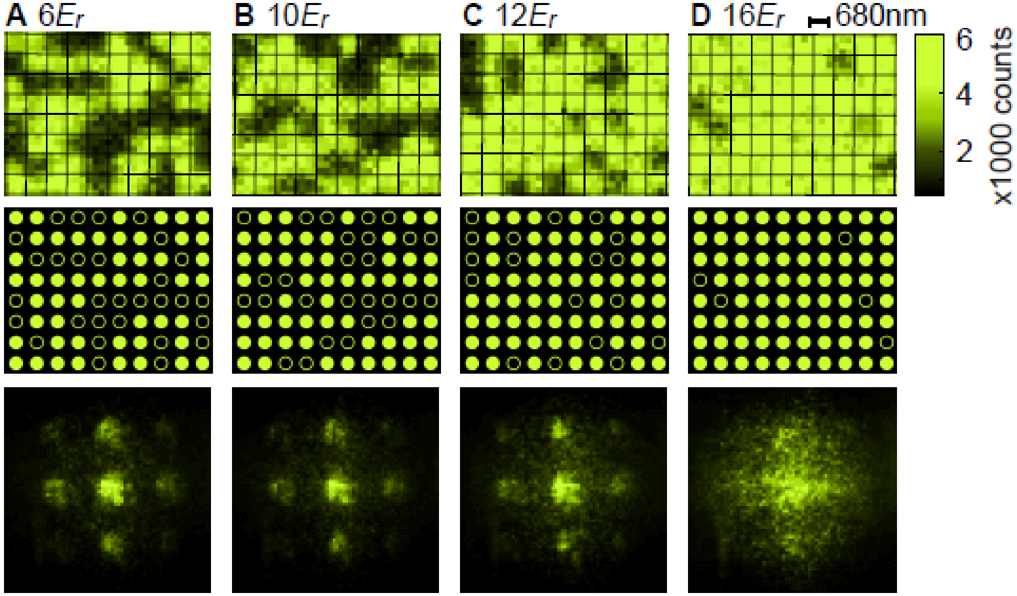
Time of flight (TOF) expansion

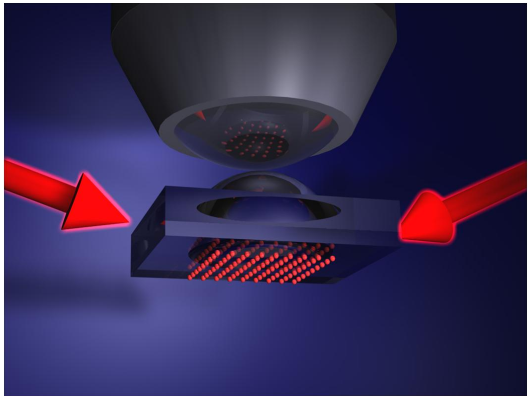
“Microscope”: amplify the spatial size

Measure the momentum distribution

Single-site measurement

<u>In-situ imaging of atom number</u>





Bakr et al. (2010)

SF

MI

Unique features of cold atom gases (3)

Time of flight (TOF) expansion

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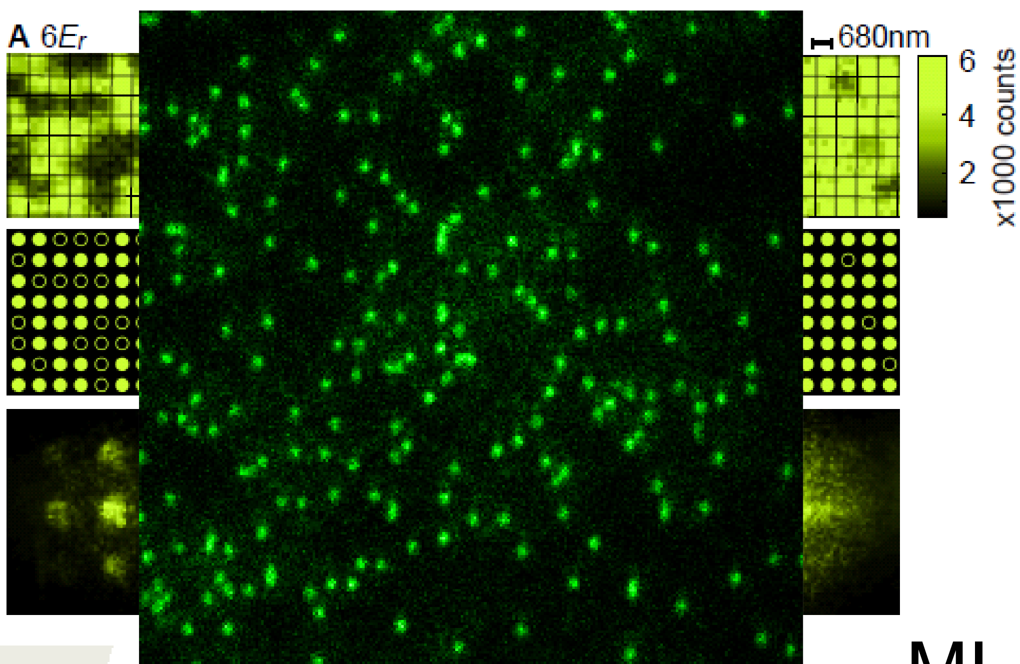
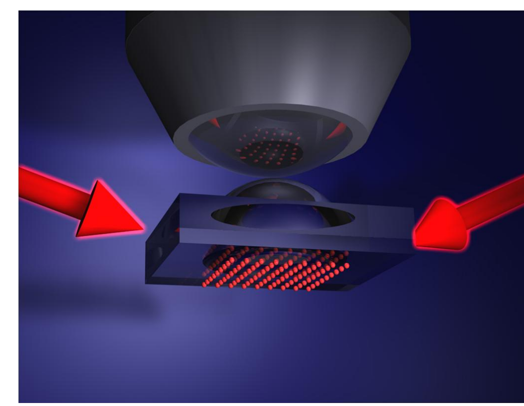


SF

MI



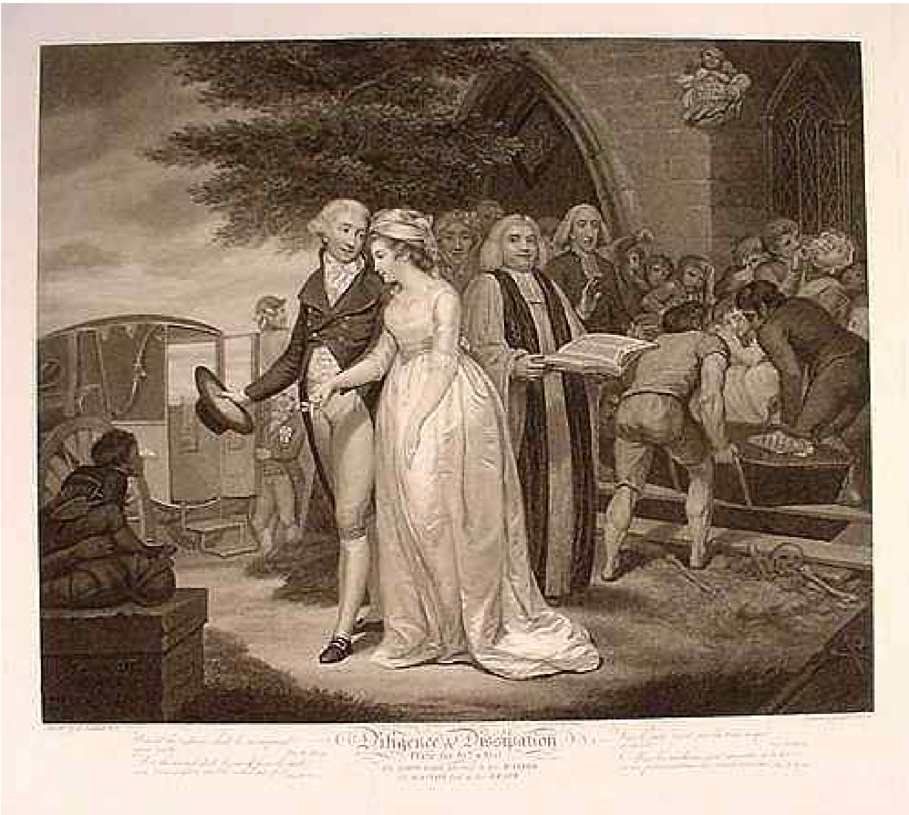
Bakr et al. (2010)

Quantum state engineering using dissipation

What is dissipation?

Dissipation (*Longman Advanced American Dictionary*)

1. *the process of making something disappear or scatter*
2. *the act of wasting money, time, energy etc.*
3. *the enjoyment of physical pleasures that are harmful to your health*



"Diligence and Dissipation" by Northcote

Dissipation: caused by coupling with environment

Dissipation in cold atom gases

Dissipation: caused by coupling with environment

- Particle losses (1-, 2-, & 3-body losses)
- Decay of atoms (de-excitation)
- Incoherent scattering of trap laser photons
- Interaction with thermal gas etc.

Escaping atoms tell where were they.

➡ Collapse of wave func.

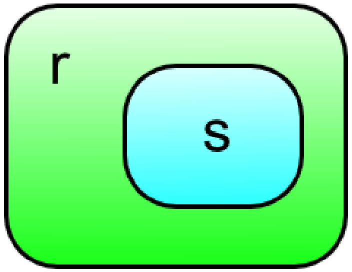
Spont. emitted photons tell the position of atoms.

➡ Localization of atoms

Usually, "dissipation" ≈ "decoherence"

But, "dissipation" ≠ "decoherence"

Quantum sys. coupled to a reservoir.



(total) = (system) + (reservoir)

ρ_{sr} : total density operator (system + reservoir)

$\rho \equiv \rho_s = \mathrm{Tr}_r[\rho_{sr}]$: system density operator

Focus on the system variables.

$$\langle \hat{A} \rangle = \mathrm{Tr}_{sr}[\rho_{sr}\hat{A}] = \mathrm{Tr}_s[\mathrm{Tr}_r[\rho_{sr}]\hat{A}] = \mathrm{Tr}_s[\underline{\rho}\hat{A}]$$

What we need is the system density operator ρ.

Master equation: EOM for the system density op. within Born-Markov approx.

$$\frac{d\rho}{dt} = -i[H,\rho] + \boxed{\frac{\gamma}{2}(2c\rho c^\dagger - c^\dagger c\rho - \rho c^\dagger c)}$$

Dissipation

c : jump op.

γ : dissipation rate

Born-Markov approx.

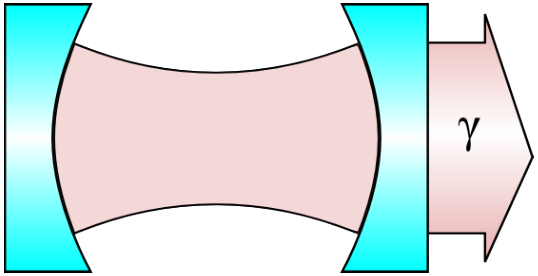
- Weak system-bath coupling (Born)
  coupling term << H or reservoir
- Short correlation time of the reservoir (Markov)
  << dynamical timescale of sys
  Reservoir has no memory.

Poor man's derivation of master eq. (1)

ex. Photons in a lossy cavity

γ: loss rate

a: annihilation op. of photons



Prob. of a photon escaping from cavity in δt.

$$\Delta P = \gamma \langle \Psi | a^\dagger a | \Psi \rangle \delta t$$

$$|\Psi\rangle \xrightarrow{\delta t} \begin{cases} |\Psi_{\mathrm{emit}}\rangle & \Delta P \\ |\Psi_{\mathrm{no\ emit}}\rangle & 1 - \Delta P \end{cases}$$

$$|\Psi_{\mathrm{emit}}\rangle = \frac{a|\Psi\rangle}{\langle \Psi | a^\dagger a | \Psi \rangle^{1/2}} = (\gamma \delta t / \Delta P)^{1/2} a |\Psi\rangle$$

$$|\Psi_{\mathrm{no\ emit}}\rangle = \frac{e^{-iH_{\mathrm{eff}}\delta t}|\Psi\rangle}{\langle \Psi | e^{iH_{\mathrm{eff}}^\dagger \delta t} e^{-iH_{\mathrm{eff}}\delta t} | \Psi \rangle^{1/2}} \simeq \frac{(1 - iH\delta t - \frac{\gamma}{2}\delta t a^\dagger a)|\Psi\rangle}{(1 - \Delta P)^{1/2}}$$

$$H_{\mathrm{eff}} = H - i\frac{\gamma}{2} a^\dagger a$$

Measurement of "no emission". $\Rightarrow$ non-Hermitian

Density operator at $t + \delta t$

$$\rho(t+\delta t) = \Delta P|\Psi_{\rm emit}\rangle\langle\Psi_{\rm emit}| + (1-\Delta P)|\Psi_{\rm no\ emit}\rangle\langle\Psi_{\rm no\ emit}|$$

$$\simeq |\Psi\rangle\langle\Psi| - i\delta t\ [H, |\Psi\rangle\langle\Psi|]$$

$$+\frac{\gamma}{2}\delta t\left(2a|\Psi\rangle\langle\Psi|a^\dagger - a^\dagger a|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|a^\dagger a\right)$$

Since $|\Psi\rangle\langle\Psi| = \rho(t)$

$$\frac{d\rho}{dt} = -i[H,\rho] + \frac{\gamma}{2}\left(\boxed{2a\rho a^\dagger}\boxed{- a^\dagger a\rho - \rho a^\dagger a}\right)$$

Quantum jump by emission

Damping by non-unitary evolution.

Jump operator = *a*

Master eq. based on the measurement theory (1)

A measurement is done in the time interval (t, $t+T$).

Measurement op. $M_\alpha(T)$ state after measurement $\propto M_\alpha(T)|\psi\rangle$

α: result of the measurement related to the measurement basis $|\alpha\rangle$.

Completeness: $$\sum_\alpha M_\alpha^\dagger(T) M_\alpha(T) = 1$$

Prob. of result α

$$P_\alpha = \mathrm{Tr}[M_\alpha(T)\rho(t)M_\alpha^\dagger(T)] \equiv \mathrm{Tr}[\tilde{\rho}_\alpha(t+T)]$$

State after the measurement conditioned by α

$$\rho_\alpha(t+T) = \tilde{\rho}_\alpha(t+T)/P_\alpha$$

Non-selective evolution

$$\rho(t+T) = \sum_\alpha P_\alpha \rho_\alpha(t+T) = \sum_\alpha M_\alpha(T)\rho(t)M_\alpha^\dagger(T)$$

Master eq. based on the measurement theory (2)

Continuous measurement: $T \to dt$

Set of measurement op. (photons in a lossy cavity)

$$
\begin{cases}
\text{Emission: } & M_1(dt) = \sqrt{\gamma dt}\, a \\
\text{No emission: } & M_0(dt) = 1 - \left(iH + \boxed{\frac{\gamma}{2} a^\dagger a} \right) dt
\end{cases}
$$

Prob. of escape.

Completeness

Non-selective evolution:

$$\rho(t+dt) = M_0(dt)\rho(t)M_0^\dagger(dt) + M_1(dt)\rho(t)M_1^\dagger(dt)$$

$$\frac{d}{dt}\rho(t) = -i[H,\rho] + \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a)$$

Motivation (1)

Kraus et al. PRA **78**, 042307 (2008).

Master eq.

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i[H,\rho] + \sum_i \frac{\gamma_i}{2}(2c_i\rho c_i^\dagger - c_i^\dagger c_i \rho - \rho c_i^\dagger c_i)$$

1. $H|\Psi\rangle = E|\Psi\rangle$
2. $\forall i \quad c_i|\Psi\rangle = 0$: dark st.
3. Uniqueness
   $|\Psi\rangle$ is the only st.;
   $\mathcal{L}(|\Psi\rangle\langle\Psi|) = 0$

target st.

$$\rho \xrightarrow[t\to\infty]{} |\Psi\rangle\langle\Psi|$$

Task: Construct a parent Liouvillian

State preparation using dissipation!

No need of dynamical manipulations. "Just wait."

Works for any initial states.

Example: Pumped single-mode field with loss

[Agarwal (1988)]

Pumping Hamiltonian:

$$H = \frac{g}{2}(a + a^\dagger)$$

Master eq.:

$$\frac{d\rho}{dt} = -i[H,\rho] + \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)$$
$$= \frac{\gamma}{2}(2c\rho c^\dagger - c^\dagger c\rho - \rho c^\dagger c) \equiv \mathcal{D}[c]\rho$$

with $c \equiv a + ig/\gamma$

$\frac{d\rho}{dt} = \mathcal{D}[c]\rho = 0$ if & only if $(a + ig/\gamma)\rho = 0 = \rho(a + ig/\gamma)^\dagger$

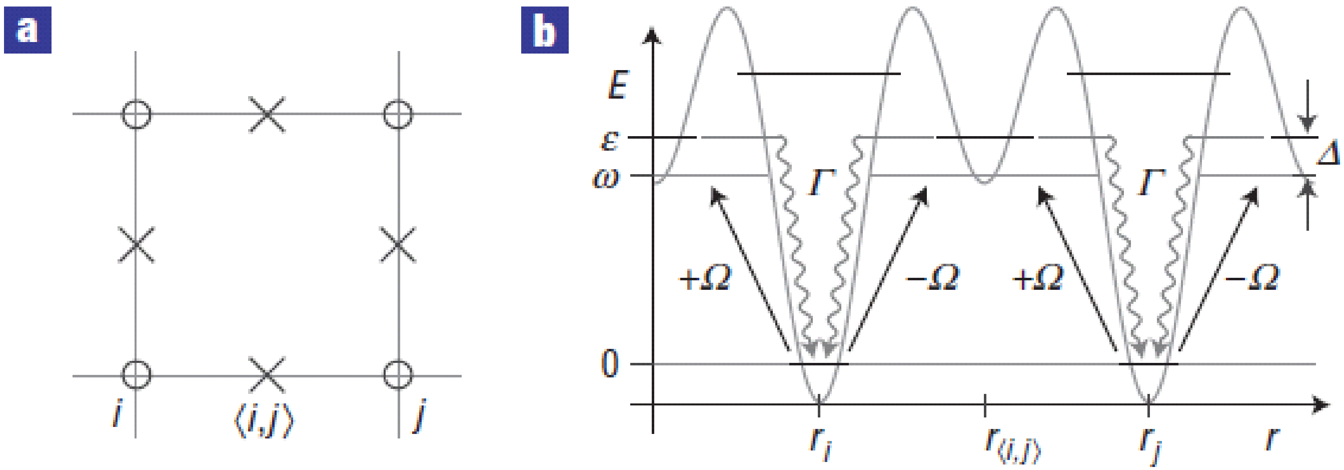
$\Rightarrow$ $\rho = |-ig/\gamma\rangle\langle -ig/\gamma|$: coherent st.

"Dissipation-induced coherence"

Diehl et al., Nature Phys. **4**, 878 (2008).

Method to prepare the SF phase
in optical lattices using dissipation.

Lattice system immersed in a BEC.



jump op. $c_{i,j} = (a_i^\dagger + a_j^\dagger)(a_i - a_j)$

$\rho \longrightarrow |\mathrm{BEC}\rangle\langle\mathrm{BEC}|$ with $|\mathrm{BEC}\rangle = \dfrac{(a^\dagger_{\mathbf{q}=0})^N}{\sqrt{N!}}|0\rangle$

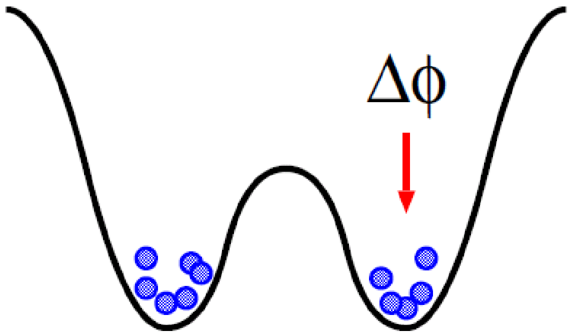
Dissipation-induced squeezing

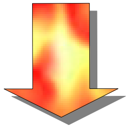
GW & Harri Mäkelä, "Dissipation-induced squeezing" arXiv:1101.4845 [cond-mat, quant-gas].

Matter-wave interf. and 2-mode sys.

Matter-wave interferometer : interferometer with cold atoms

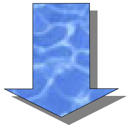
Ex. cold Bose gases in a double-well pot.





Release from the trap.

Expand & interfere



Measure the force field, etc.
from the interference pattern.

2-site Bose-Hubbard Hamiltonian

Many body Hamiltonian of Bose gases

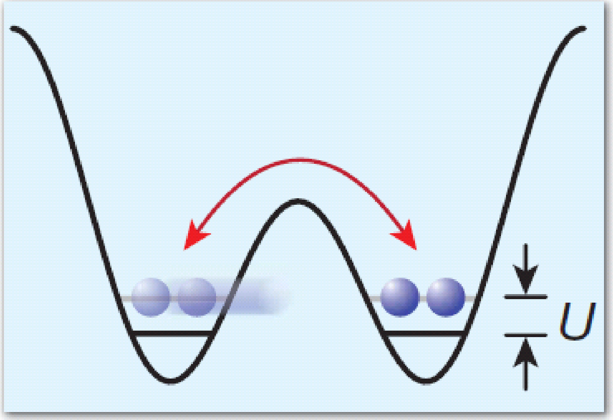
$$\hat{H}(t) = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} \nabla \hat{\psi}^\dagger \cdot \nabla \hat{\psi} + \hat{\psi}^\dagger V(\mathbf{r}, t) \hat{\psi} + \frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right]$$

Two-mode approx. $\hat{\psi} = u_L(\mathbf{r}) \hat{c}_L + u_R(\mathbf{r}) \hat{c}_R$

Valid provided $\hbar\omega_{\text{well}} \gg gn \Rightarrow d_{\text{well}}/a_s \gg N$

2-site Bose-Hubbard Hamiltonian

$$\hat{H} = -J(\hat{c}_R^\dagger \hat{c}_L + \hat{c}_L^\dagger \hat{c}_R) + \frac{U}{2}(\hat{c}_R^\dagger \hat{c}_R^\dagger \hat{c}_R \hat{c}_R + \hat{c}_L^\dagger \hat{c}_L^\dagger \hat{c}_L \hat{c}_L)$$



hopping matrix element

$$J \sim -\int d\mathbf{r} \left[\frac{\hbar^2}{2m} (\nabla u_L^* \cdot \nabla u_R) + u_L^* V u_R \right]$$

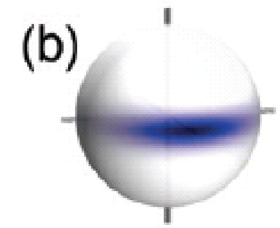
on-site interaction

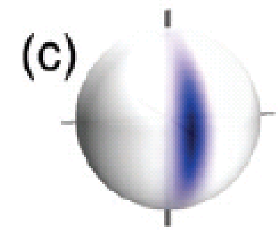
$$U \sim g \int d\mathbf{r}\ u_{R,L}^4$$

Squeezed st. in matter-wave interf.

Uncertainty relation $\Delta X_1 \Delta X_2 \lesssim \hbar$

Squeezing: decreasing uncertainty of one variable
(in expense of increasing the other's)

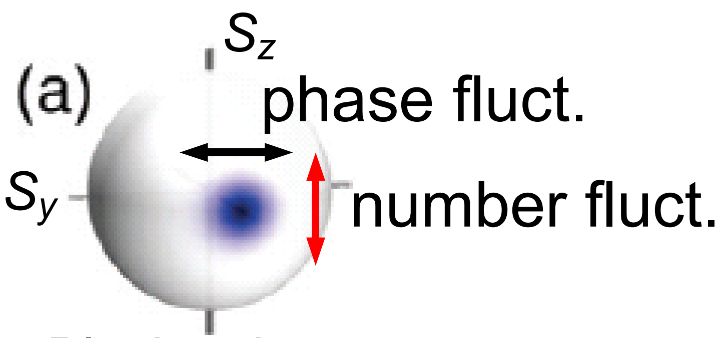




Two-mode case

Number squeezing $\xi_N \equiv \langle \Delta S_z^2 \rangle^{1/2} / \sqrt{N/2} < 1$

Phase squeezing $\xi_{\mathrm{phase}} \equiv \langle \Delta S_y^2 \rangle^{1/2} / \sqrt{N/2} < 1$



Bloch sphere

$$S_x = (a_1^\dagger a_2 + a_2^\dagger a_1)/2$$

$$S_y = (a_1^\dagger a_2 - a_2^\dagger a_1)/2i$$

$$S_z = (a_1^\dagger a_1 - a_2^\dagger a_2)/2$$

Phase sq. st.: High visibility of interf. pattern.
Good for readout.

(Number sq. st.: Small phase diffusion.
Longer phase accum. time.)

Take-home messages

GW & Mäkelä, arXiv:1101.4845.

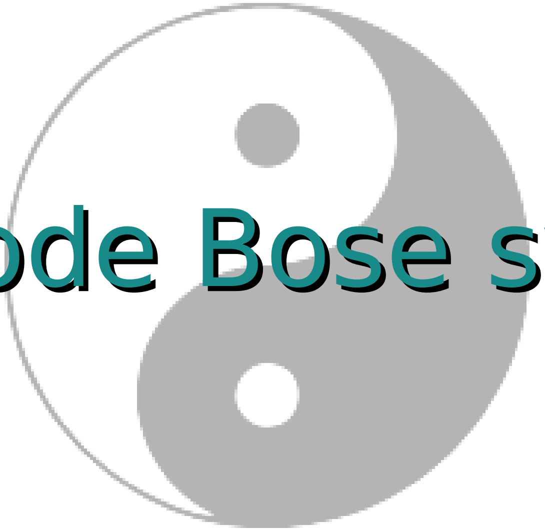
A method to create number & phase sq. st. in any 2-mode Bose sys. using dissipation.

Perform number/phase squeezing/anti-squeezing in a controllable manner.

Extension to optical lattices

- Control of the phase boundaries in a steady-state phase diagram.
- New phase characterized by non-zero cond. fraction and thermal-like particle statistics.

Two-mode Bose systems



Squeezing jump operator

Coherent st.: $|\phi\rangle \propto (e^{i\phi/2}a_1^\dagger + e^{-i\phi/2}a_2^\dagger)^N|0\rangle$

Squeezing jump op.

$$c = \boxed{(a_1^\dagger + a_2^\dagger)(a_1 - a_2)} + \epsilon\boxed{(a_1^\dagger - a_2^\dagger)(a_1 + a_2)} \quad (-1 < \epsilon < 1)$$

$|\phi = 0\rangle \propto (a_1^\dagger + a_2^\dagger)^N|0\rangle$

is a dark st.

$\because \; [a_1 - a_2, a_1^\dagger + a_2^\dagger] = 0$

$|\phi = \pi\rangle \propto (a_1^\dagger - a_2^\dagger)^N|0\rangle$

is a dark st.

$\because \; [a_1 + a_2, a_1^\dagger - a_2^\dagger] = 0$

$$c = 2(1+\epsilon)S_z - 2i(1-\epsilon)S_y$$

when ε =1, $c = 4S_z \propto \Delta N$

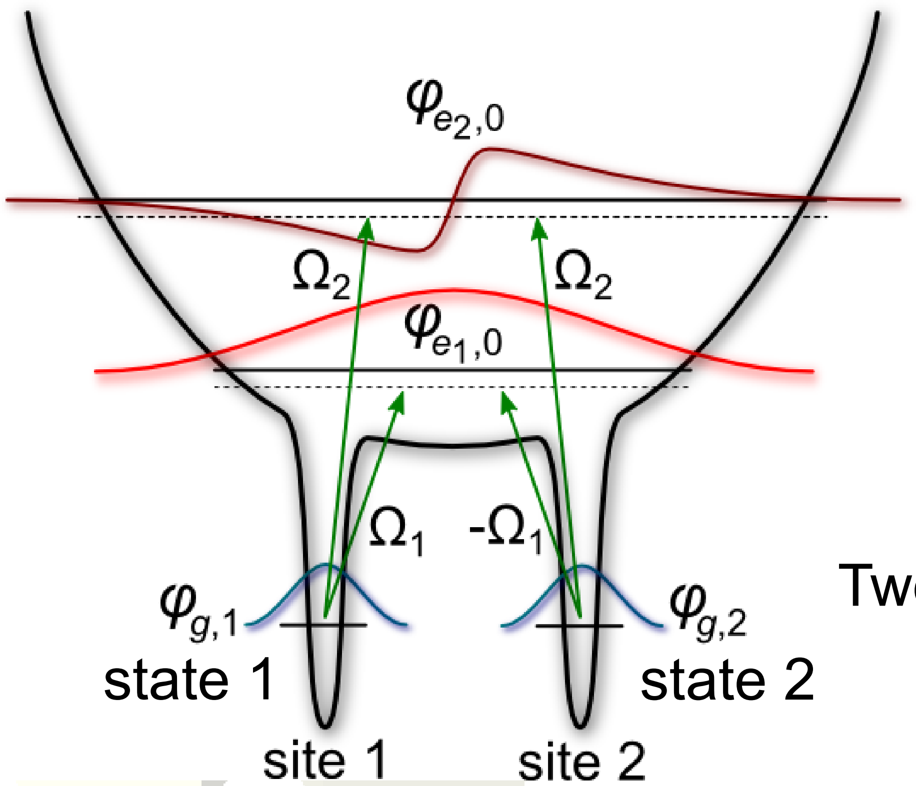
Dark st. is a Fock st. with ΔN=0.

Setup: $\boxed{\text{trapped atoms}}$ + $\boxed{\text{background BEC}}$

system — reservoir of s.f. phonons

・States 1& 2 are Raman coupled to 2 excited states with even & odd parity.

・Atoms in the excited states decay into states 1 & 2 through s.f. phonon emission.



$$c = \underbrace{\boxed{(a_1^\dagger + a_2^\dagger)(a_1 - a_2)}}_{c_1} + \epsilon \underbrace{\boxed{(a_1^\dagger - a_2^\dagger)(a_1 + a_2)}}_{c_2}$$

→ c_2

→ c_1

$$\epsilon = -\frac{\Omega_2}{\Omega_1}\frac{\Delta_1}{\Delta_2}\frac{\phi_{e_2,0}(x_0)}{\phi_{e_1,0}(x_0)}$$

Ω_i : effective Rabi freq.

Δ_i : detuning

Two excited states are adiabatically eliminated.

Coherent st. analysis (valid for N>>1)

$$\langle S_x^2 \rangle \simeq \langle S_x \rangle^2 \quad \text{with} \quad \langle S_x \rangle \simeq N/2 + O(N^0)$$

$$\frac{d}{dt}\langle S_{y,z}^2 \rangle = \mathrm{Tr}\left[\dot{\rho} S_{y,z}^2\right]$$

$$\langle S_i S_j S_k \rangle \simeq \langle S_i S_j \rangle \langle S_k \rangle + \langle S_i \rangle \langle S_j S_k \rangle + \langle S_i S_k \rangle \langle S_j \rangle - 2\langle S_i \rangle \langle S_j \rangle \langle S_k \rangle$$

master eq. ⇒ $$\frac{d}{dt}\langle S_{y,z}^2 \rangle \simeq -4N\gamma(1-\epsilon^2)\langle S_{y,z}^2 \rangle + N^2\gamma(1 \pm \epsilon)^2$$

In the steady state

Number squeezing param.: $$\xi_N = \frac{\langle S_z^2 \rangle^{1/2}}{\sqrt{N/2}} \simeq \sqrt{\frac{1-\epsilon}{1+\epsilon}}$$

Phase squeezing param.: $$\xi_{\mathrm{phase}} = \frac{\langle S_y^2 \rangle^{1/2}}{\sqrt{N/2}} \simeq \sqrt{\frac{1+\epsilon}{1-\epsilon}}$$

ε > 0 ⇒ number sq. & phase anti-sq.

ε < 0 ⇒ phase sq. & number anti-sq.

Squeezing parameter and purity *P*[*ρ*]

purity $P[\rho] \equiv \{(N+1)\mathrm{Tr}[\rho^2]-1\}/N = \begin{cases} 1 & \text{(pure st.)} \\ 0 & \text{(completely mixed st.)} \end{cases}$

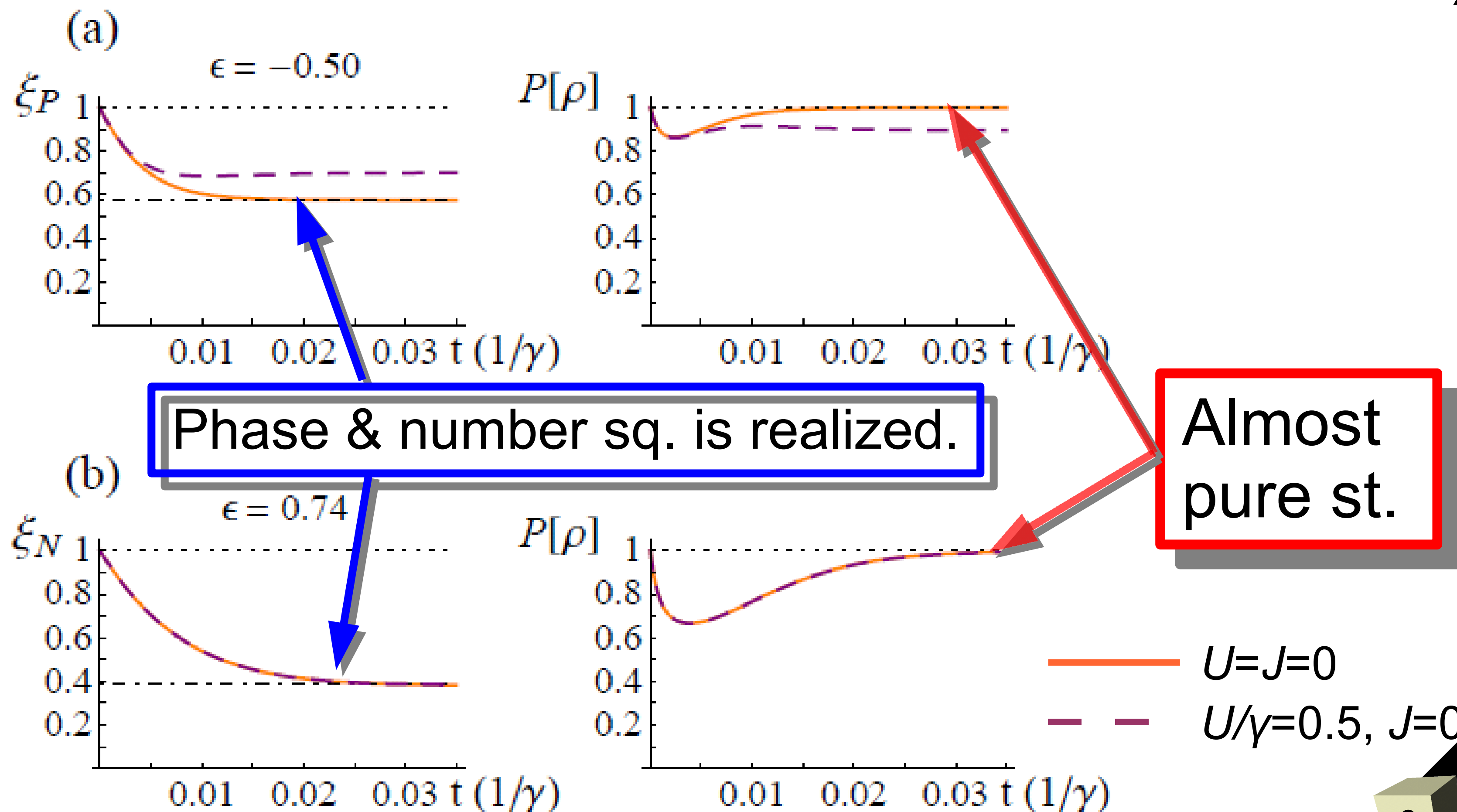
Time evolution for *N*=100

ε < 0

Phase squeezing

ε > 0

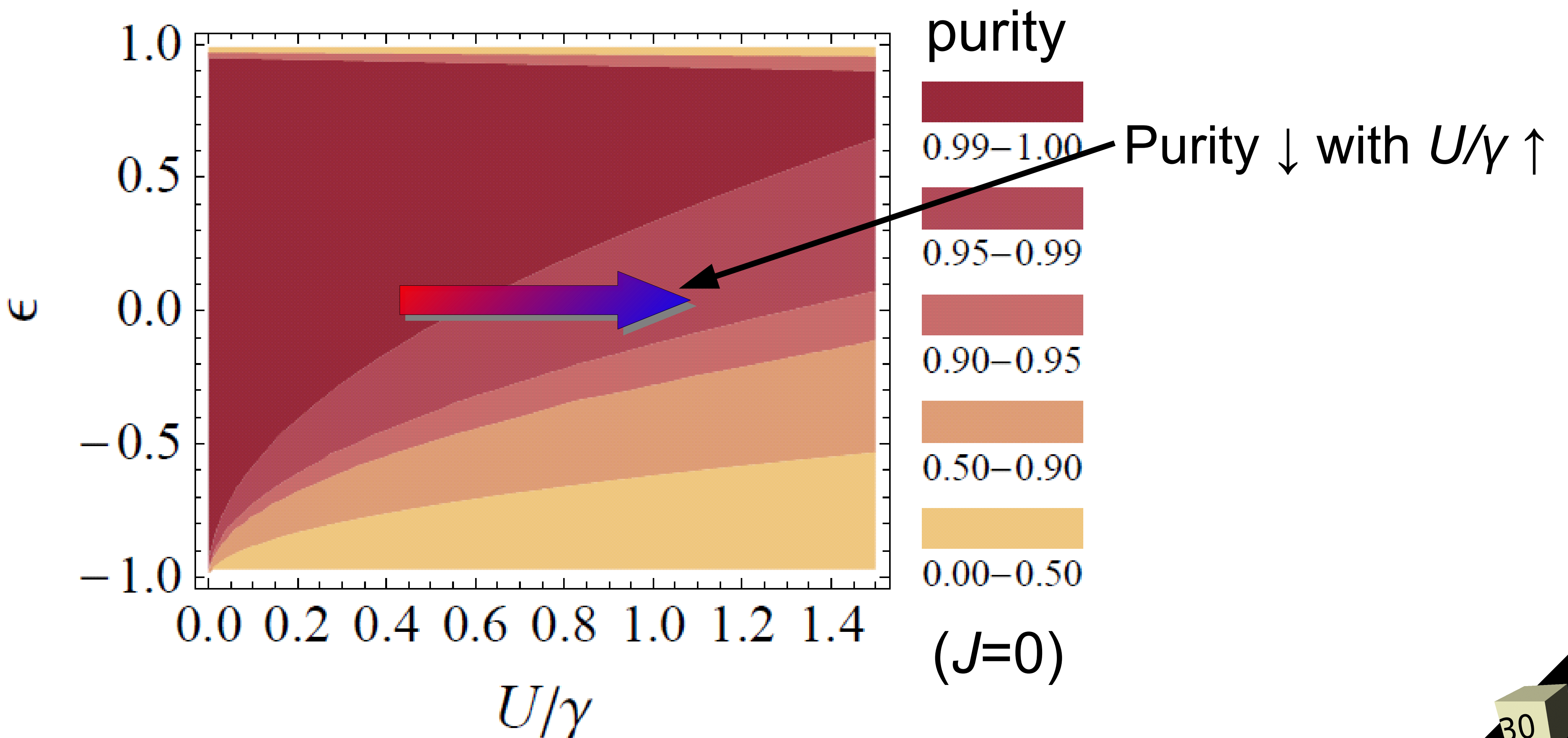
Number squeezing



2-site BH Hamiltonian: $H = -2JS_x + US_z^2$

S_x : eigenst. are coherent st.
S_z : eigenst. are Fock st.

Nonzero-*U* reduces purity.



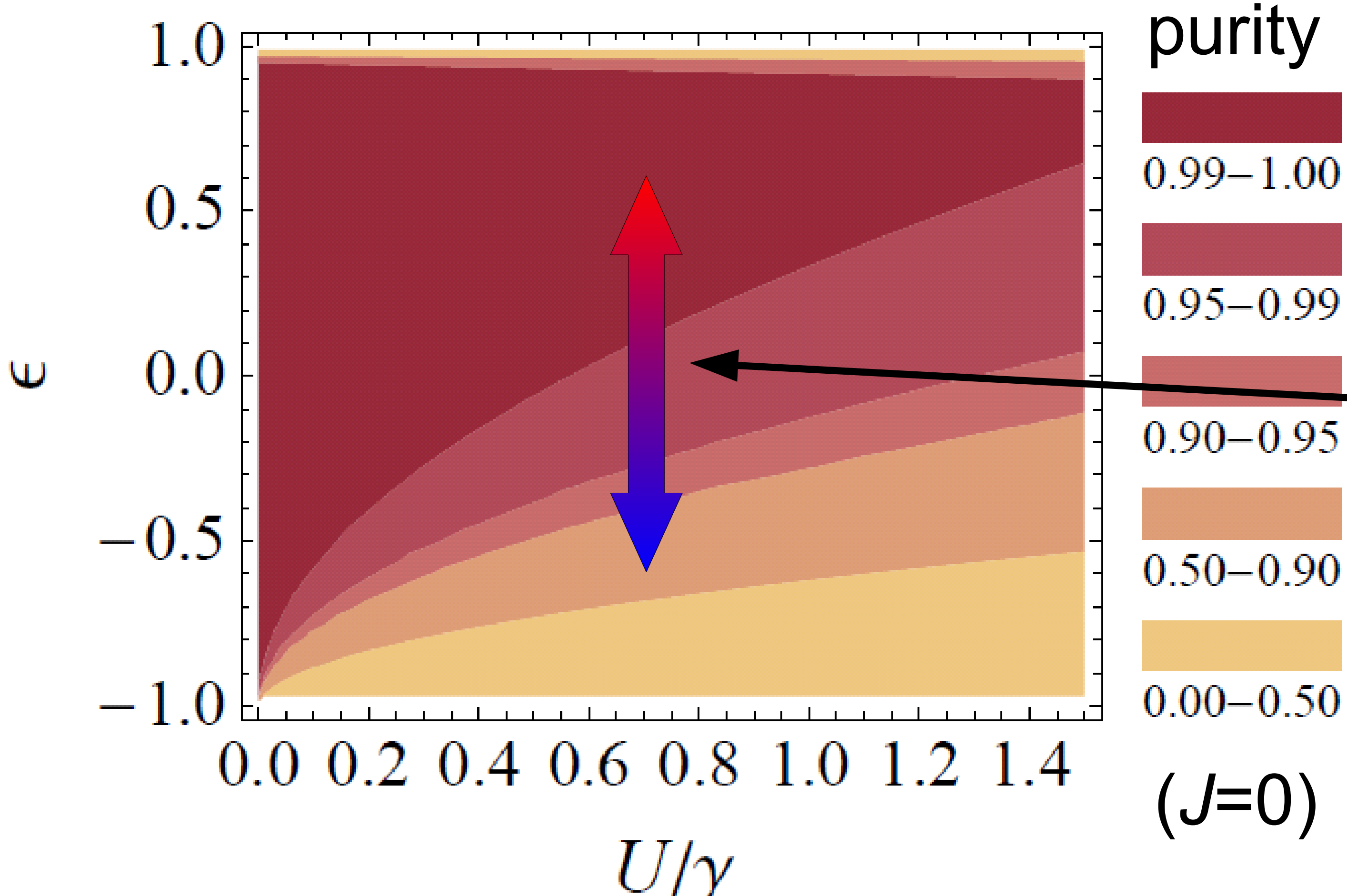
Purity of steady st. with BH Hamiltonian

2-site BH Hamiltonian: $H = -2JS_x + US_z^2$

S_x : eigenst. are coherent st.

S_z : eigenst. are Fock st.

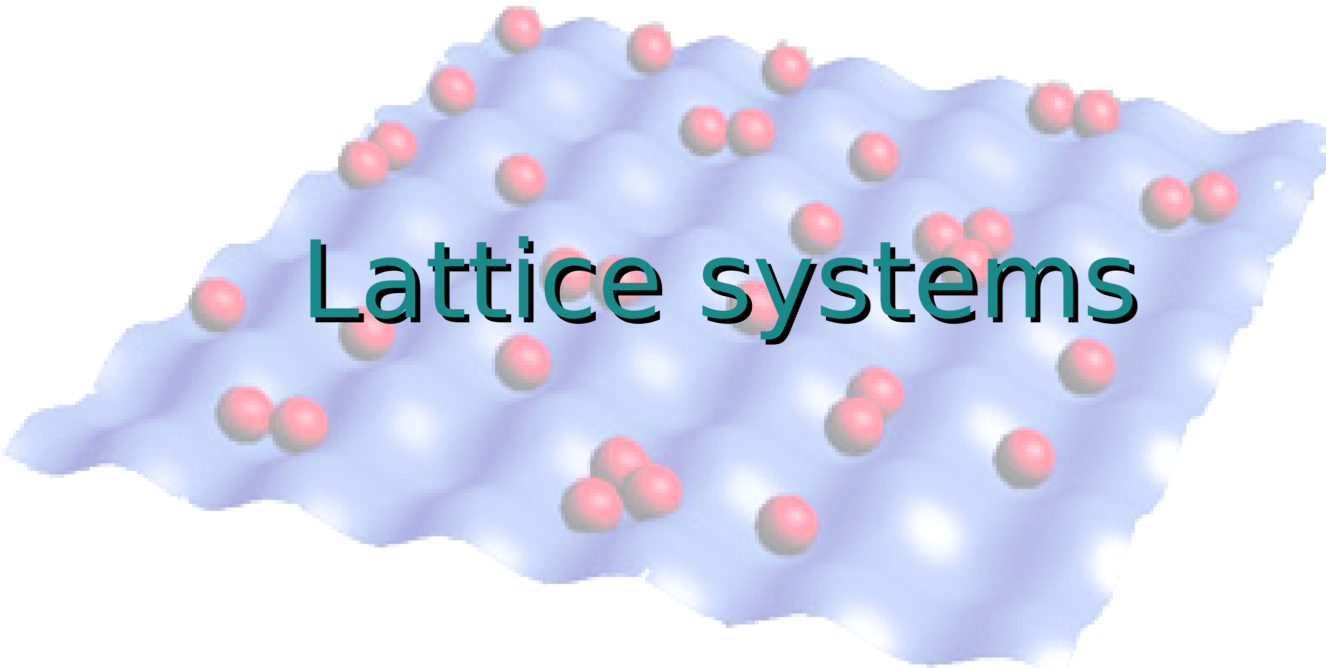
Nonzero-*U* reduces purity.



Purity ↓ with U/γ ↑

Purity is higher for $\varepsilon > 0$ (number sq.)

Lattice systems



Optical trap

Energy shift $$\Delta E = -\frac{1}{2}\alpha\langle E^2\rangle_t$$ α: polarizability

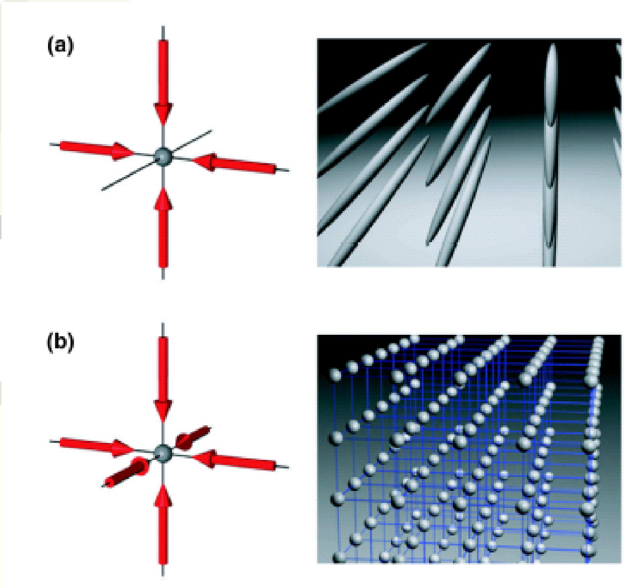
Spatially varying *E* field of lasers

→ **Atoms can be trapped**

Standing wave by two counter propagating lasers

→ **Periodic potential (“optical lattice”)**

<u>Various dimensions</u>



$$E(z,t) = E_0\cos(qz-\omega t) + E_0\cos(-qz-\omega t)$$
$$= 2E_0\cos qz\cos\omega t$$

→ $$\langle E(z,t)^2\rangle_t = 2E_0^2\cos^2 qz$$

Lattice const. *d=π/q*

Squeezing jump operator on lattices

Jump op. acting on site *i* and *j*.

$$c_{ij} = (a_i^\dagger + a_j^\dagger)(a_i - a_j) + \epsilon(a_i^\dagger - a_j^\dagger)(a_i + a_j)$$

Master eq.

$$\partial_t \rho = -i[H,\rho] + \boxed{\frac{\gamma}{2}\sum_{\langle i,j\rangle}(2c_{ij}\rho c_{ij}^\dagger - c_{ij}^\dagger c_{ij}\rho - \rho c_{ij}^\dagger c_{ij})}$$

Drive the system into a squeezed st.

BH Hamiltonian

$$H = \boxed{-J\sum_{\langle i,j\rangle}(a_i^\dagger a_j + a_j^\dagger a_i)} + \boxed{\frac{U}{2}\sum_i n_i(n_i - 1)} - \mu\sum_i n_i$$

Delocalizes atoms.

Builds coherence.

Localizes atoms.

Destroys coherence.

Competition btw. **dissipative** term and **interaction** term.

Generalized MF Gutzwiller approach

Solve the master eq. within
generalized MF Gutzwiller approx.

[Diehl et al., PRL **105**, 015702 (2010)]

Product ansatz

$$\rho = \bigotimes_i \rho_i \quad \text{with} \quad \rho_i \equiv \mathrm{Tr}_{\neq i}[\rho]$$

(reduced density op. for site *i*)

Site-decoupling MF approx.

$$H = \sum_i h_i$$

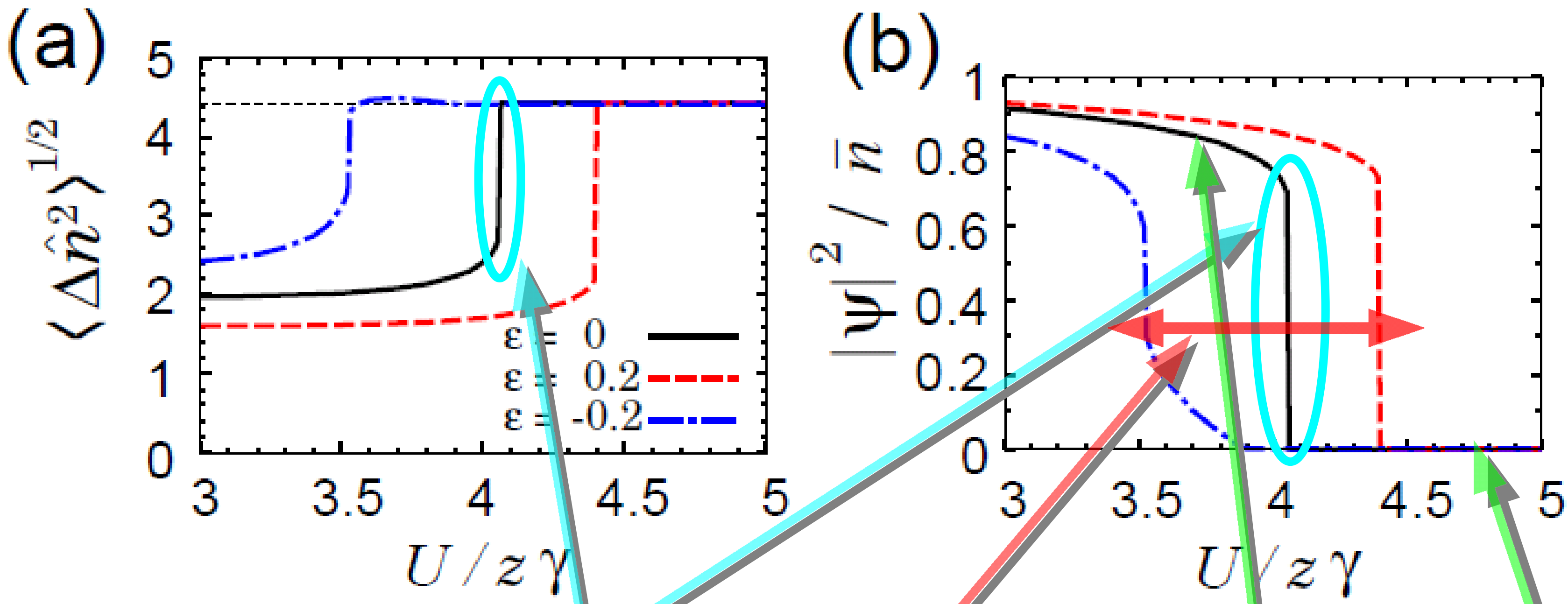
with $h_i = -J \sum_{\langle i' | i \rangle} (\langle a_{i'} \rangle a_i^\dagger + \langle a_{i'}^\dagger \rangle a_i) + \frac{U}{2} n_i (n_i - 1) - \mu n_i$

Good approx. for local properties
in higher dimensions & at larger filling factors.

Results: number fluct. & cond. frac.

Number fluctuation & condensate fraction of steady st.

filling factor $n = 4$ $J/\gamma = 1$



First order-like transition

cond. phase

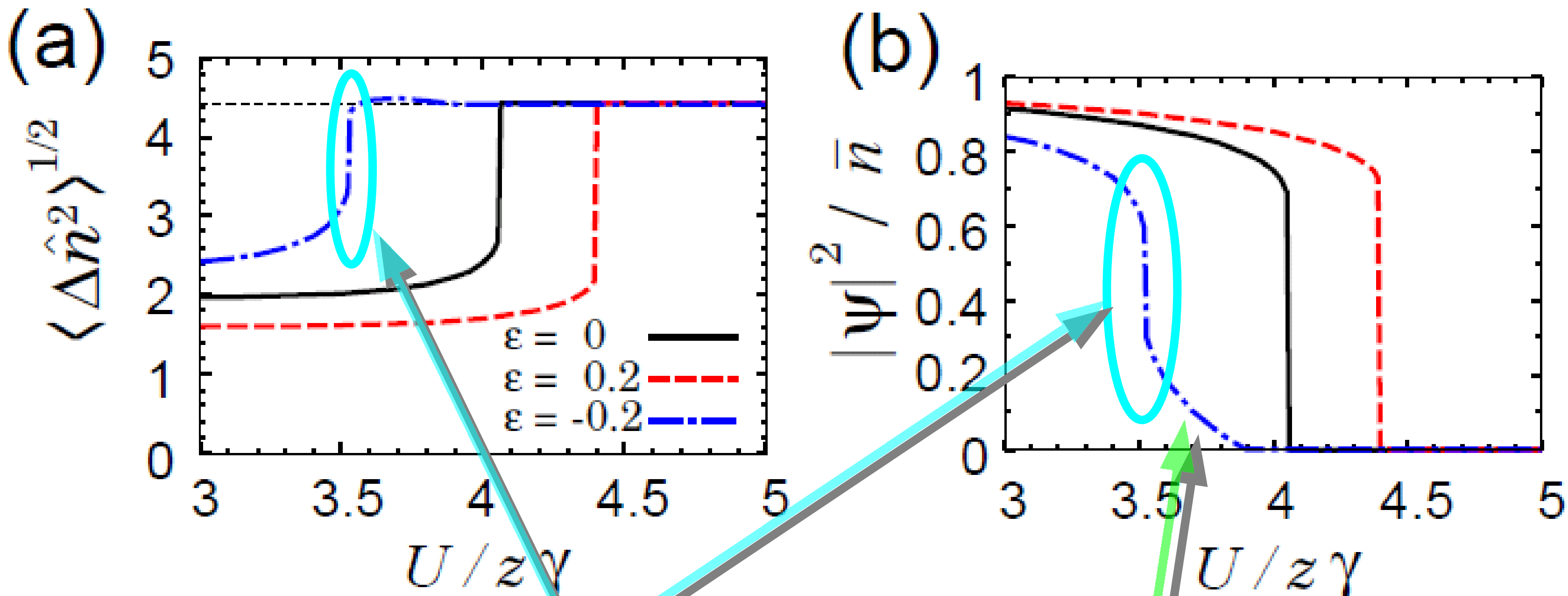
non-cond. phase

Parameter ε control the phase boundary.

$\varepsilon > 0$: expand the cond. phase

Number fluctuation & condensate fraction of steady st.

filling factor $n = 4$ $J/\gamma = 1$



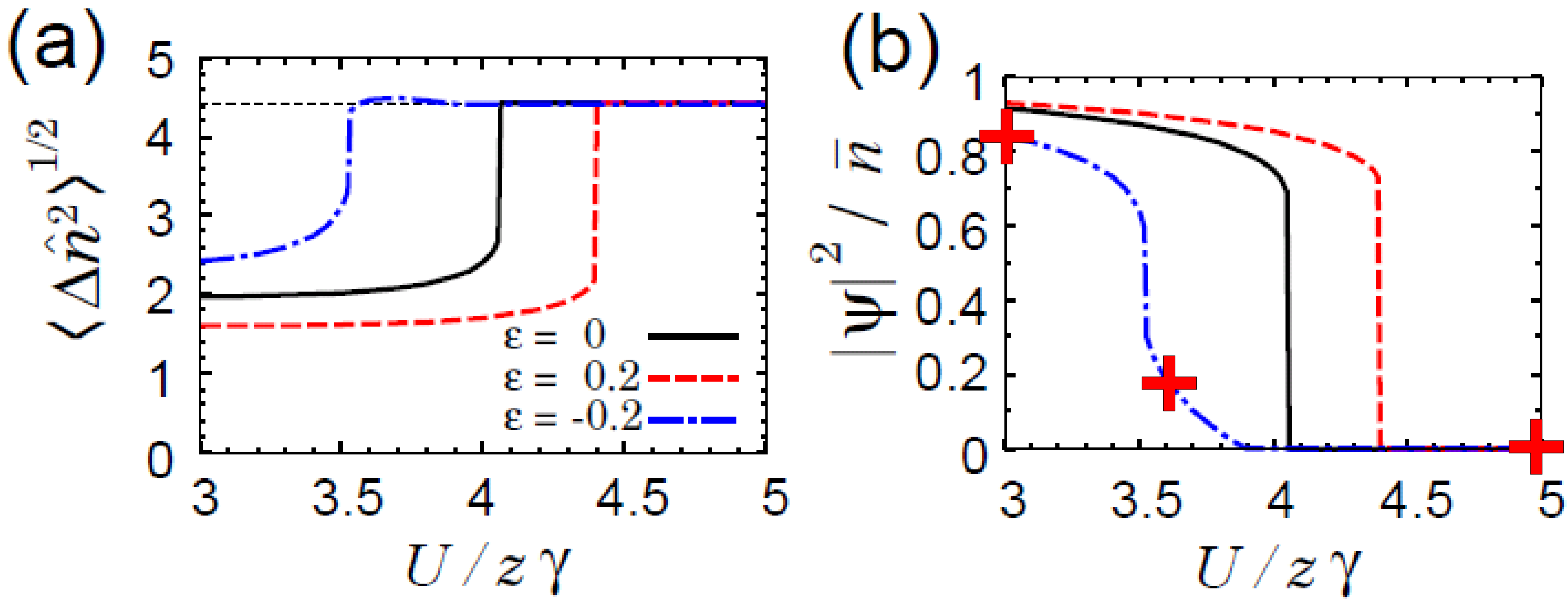
First order-like transition

Non-zero cond. fraction

Results: number fluct. & cond. frac.

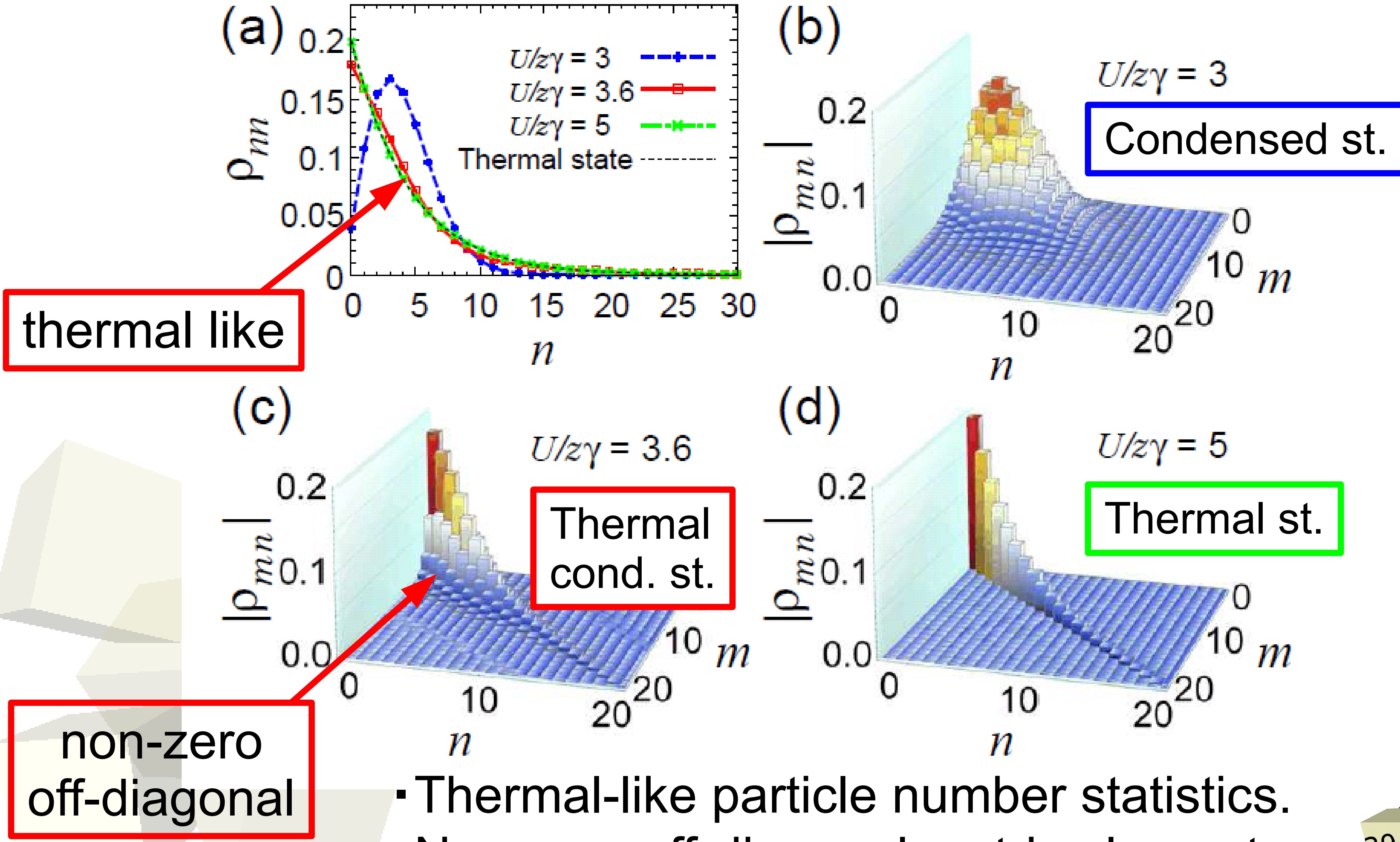
Number fluctuation & condensate fraction of steady st.

filling factor $n = 4$ $J/\gamma = 1$



Results: density matrices

Particle number statistics & density matrix

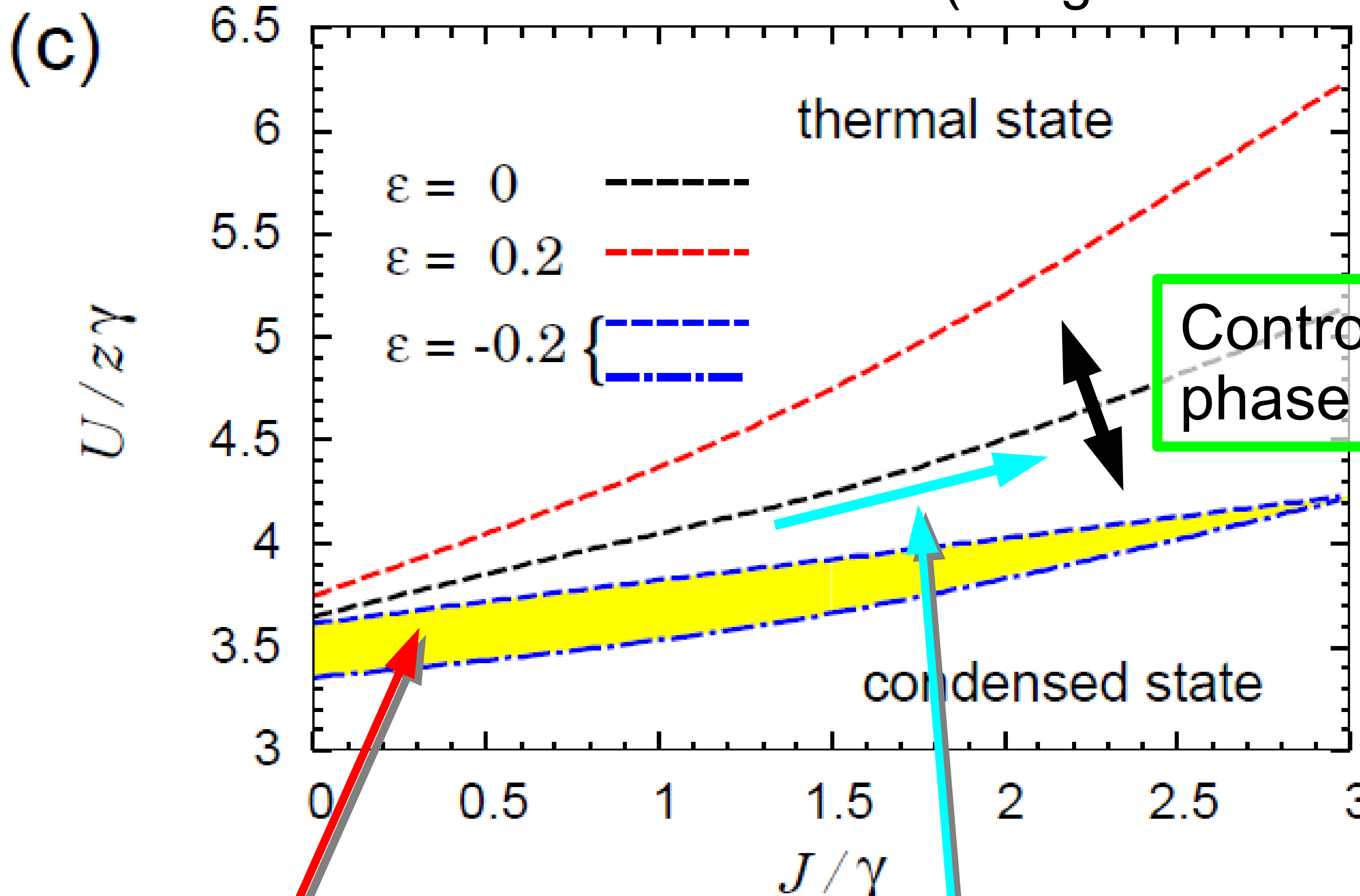


- Thermal-like particle number statistics.
- Non-zero off-diagonal matrix elements.

Results: steady state phase diagram

Non-equilibrium steady st. phase diagram

(filling factor n=4)



Control of the phase boundary.

Thermal condensed phase

Bigger cond. st. region for large *J*.
(*J*-term builds the coherence.)

Summary & conclusion

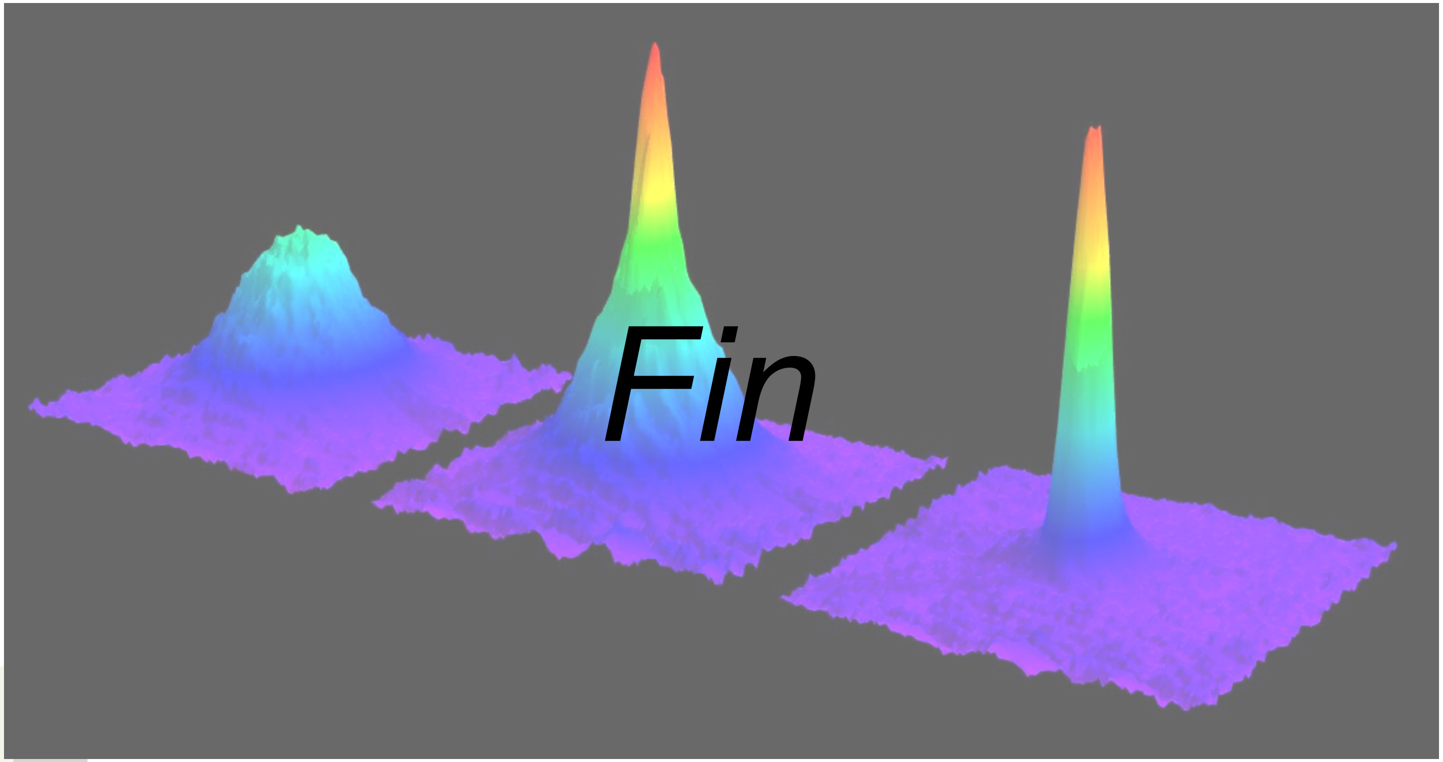
Quantum state engineering using dissipation

Dissipation can build quantum coherence.
State preparation using dissipation is possible.

GW & Mäkelä, arXiv:1101.4845.

We found a method to create number & phase sq. st. in any 2-mode Bose sys. using dissipation.

- Create number/phase squeezing/anti-squeezing st. in a controllable manner.
- Extension to optical lattices gives control of the phase boundaries in steady-st. phase diagram.
- "Thermal condensed phase":
  A new phase characterized by non-zero cond. fraction & thermal-like particle statistics.



Thank you for your attention.

Typical scales

Cold atom gases are ultracold and ultradilute

"Ultradilute": (particle separation) >> (range of atomic pot.)

(~10 a_0)

Interparticle distance r_s ~ 100nm

Density n ~ 10^12 – 10^15 cm^{-3}

$$k_B T_c \sim \frac{\hbar^2}{2mr_s^2} \sim 1\mu\mathrm{K}$$

Coldness & diluteness

→ Atom-atom int.: Low-energy & two-body scattering

All we need is s-wave scattering length a_s.

Typically, a_s ~ 100 a_0, but tunable.

Exact form of the steady state

$$c = 4\sqrt{\epsilon}\; e^{\chi S_x} S_z e^{-\chi S_x}$$

$$\chi \equiv \mathrm{arctanh}\left[\tfrac{1-\epsilon}{1+\epsilon}\right]$$

For even *N*

Eigenst. of *c* with zero eigenvalue

$$\phi_{\mathrm{sq}} \propto e^{\chi S_x}|N/2\rangle$$

$$= \sum_{n=-N/2}^{N/2} \alpha_n |N/2-n\rangle$$

$$\alpha_n = \binom{N}{N/2}^{1/2}\binom{N}{N/2+n}^{-1/2}\left(\frac{1+\sqrt{\epsilon}}{2\epsilon^{1/4}}\right)^N \sum_{s=|n|}^{N/2}\binom{N/2}{s}\binom{N/2}{s+n}\left(\frac{1-\sqrt{\epsilon}}{1+\sqrt{\epsilon}}\right)^{2s+n}$$

For odd *N*

Non-zero elements of $c\phi_{\mathrm{sq}} \sim \epsilon^{(N+1)/2}$

ϕ_{sq} is a dark st. for *N*→∞.

Density operator

$$\rho_{\rm th} = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}} \right)^n |n\rangle\langle n|$$

$$\bar{n} \equiv \langle n \rangle = \mathrm{Tr}[\rho_{\rm th} n]$$

$$\mathrm{Tr}[\rho_{\rm th}^2] = \frac{1}{1+2\bar{n}}$$

$$\langle n^2 \rangle = \bar{n}(1+2\bar{n})$$

von Neumann entropy

$$S(\rho_{\rm th}) = (1+\bar{n}) \ln(1+\bar{n}) - \bar{n} \ln \bar{n}$$

Coincides with the thermodynamic entropy.