Symmetry protected topological phases: realization in spin chains and spin ladders

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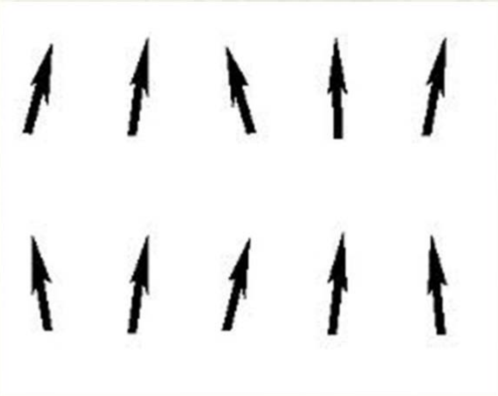
Reference : arXiv:1101.5680 (to appear on PRB); arXiv:1105.6021

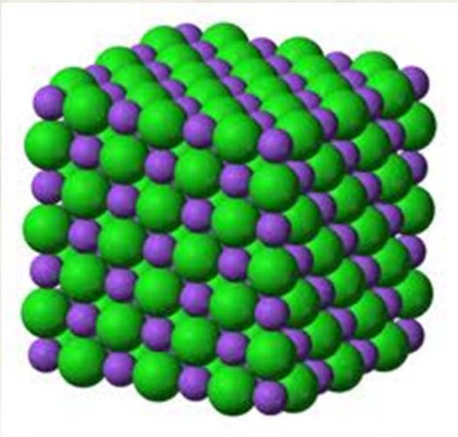
Outline

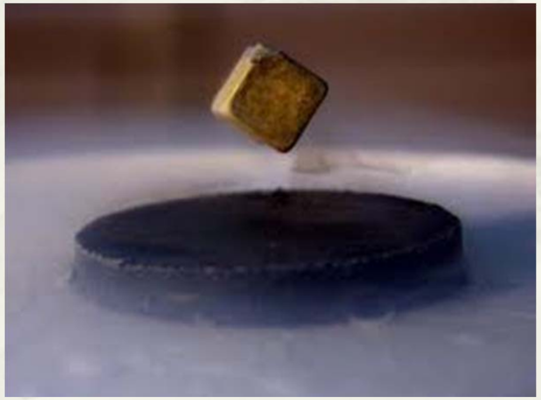
- Introduction to topological orders/Symmetry Protected Topological orders
- Classification of 1D SPT orders (focusing on D_2+T symmetry) and their realization in $S=1$ spin chains/ladders
- How to measure the SPT orders experimentally

Phases of matter: different orders (I)

- Symmetry-breaking orders (Landau)
  - Magnets: rotation symmetry breaking
  - Solid: translation symmetry breaking
  - Superconductor: U(1) symmetry breaking
  - …





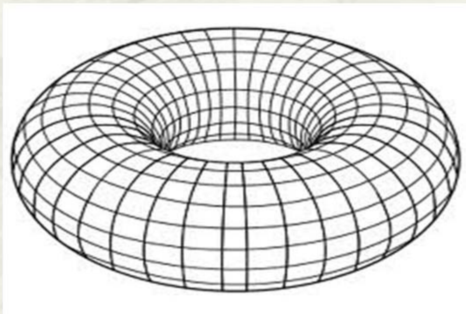


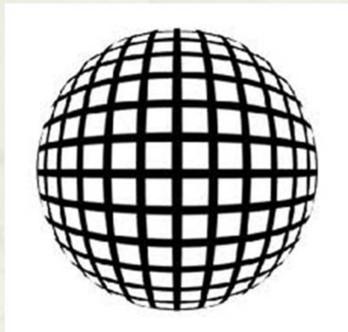
Different phases have different symmetry

Phases of matter: different orders (II)

X.-G. Wen, Phys. Rev. B 40, 7387 (1989); Int. J. Mod. Phys. B 4, 239 (1990).

- Topological order (intrinsic)
  - Non-symmetry-breaking
  - Edge states, ground state degeneracy, fractional excitations...
  - Close relation with the topology (of the manifold, group,…)





  - Long-range entanglement
  - *Examples:*
    - *Fractional Quantum Hall states*
    - *Chiral spin liquid states*
    - *String-net condensate states*

No topological order in 1D without symmetry

- All 1D gapped states are short-range entangled
- All 1D gapped states can continuously deforms into direct product states
- No phase transition between all 1D gapped states

Without symmetry,

There is only one gaped phase in 1D.

Chen, Gu, Wen, Phys. Rev. B 83, 035107 (2011)

With symmetry

- Definition of phase and phase transition
  - All the states in the same phase can be continuously transformed into each other by (symmetric) Local Unitary transformations
  - Different states with the same symmetry may belong to different phases
  - When symmetry is absent, the difference between different symmetric phases disappears

Symmetry protected topological order

(No intrinsic topological order in 1D)

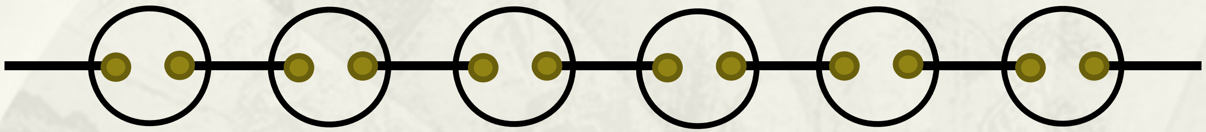
Chen, Gu, Wen, Phys. Rev. B 82, 155138 (2010)

Example: SPT order in 1D

- *S=1* model with parity, and R_x, R_yT, R_zT symmetry

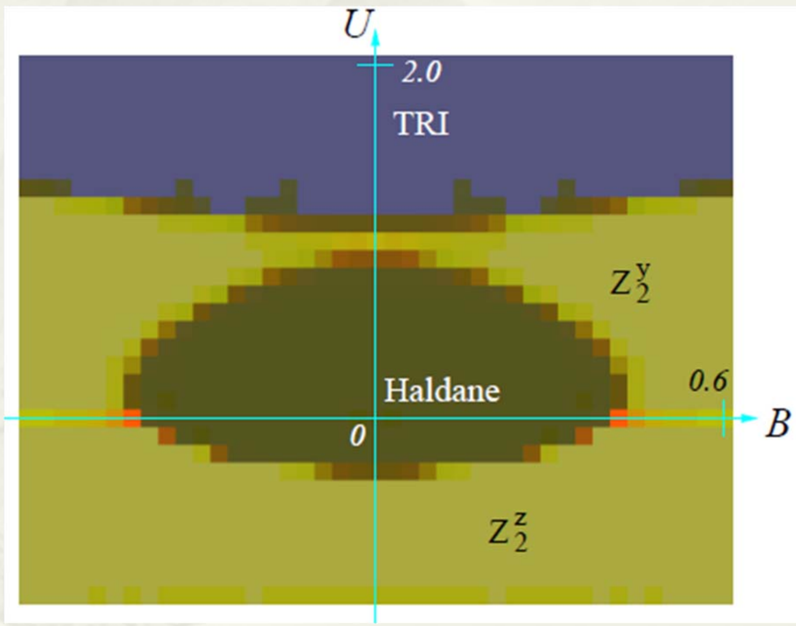
$$H = \sum_i [S_i \cdot S_{i+1} + U(S_i^z)^2 + BS_i^x]$$

  - Haldane phase (small *U,B*)



  - Large *U* (trivial) phase

$U \to \infty$: $\Psi = |0,0,\ldots\ldots 0>$, $S^z|0>=0$



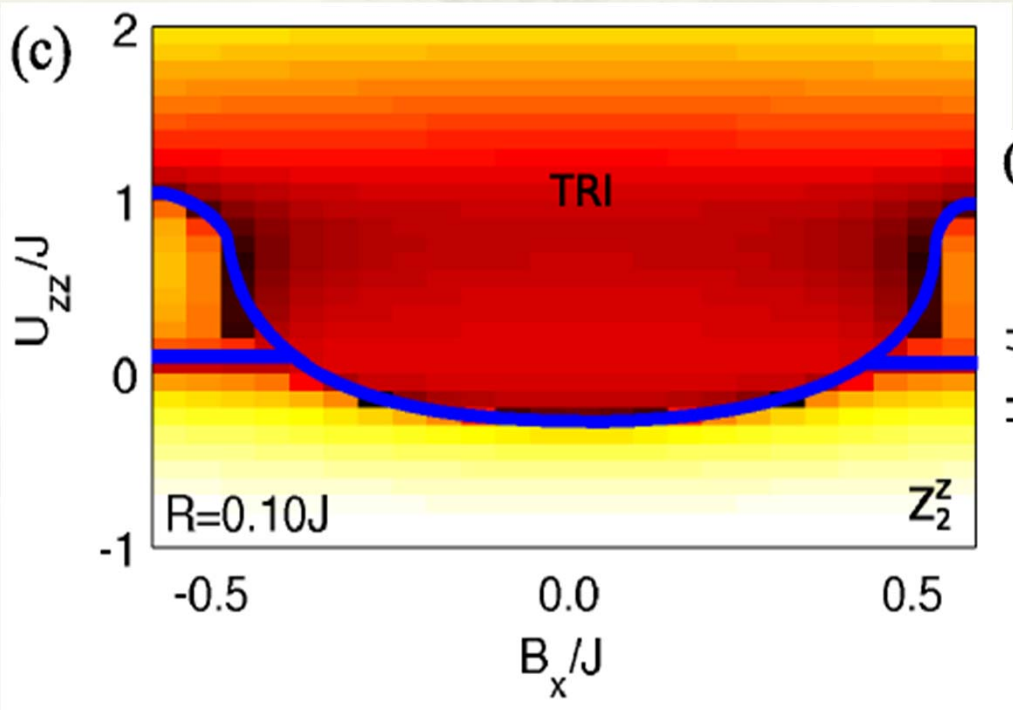
Above two phases have the same symmetry, they are distinguished by SPT order (or their different **end states**).

Gu, Wen, PRB 80, 155131 (2009)

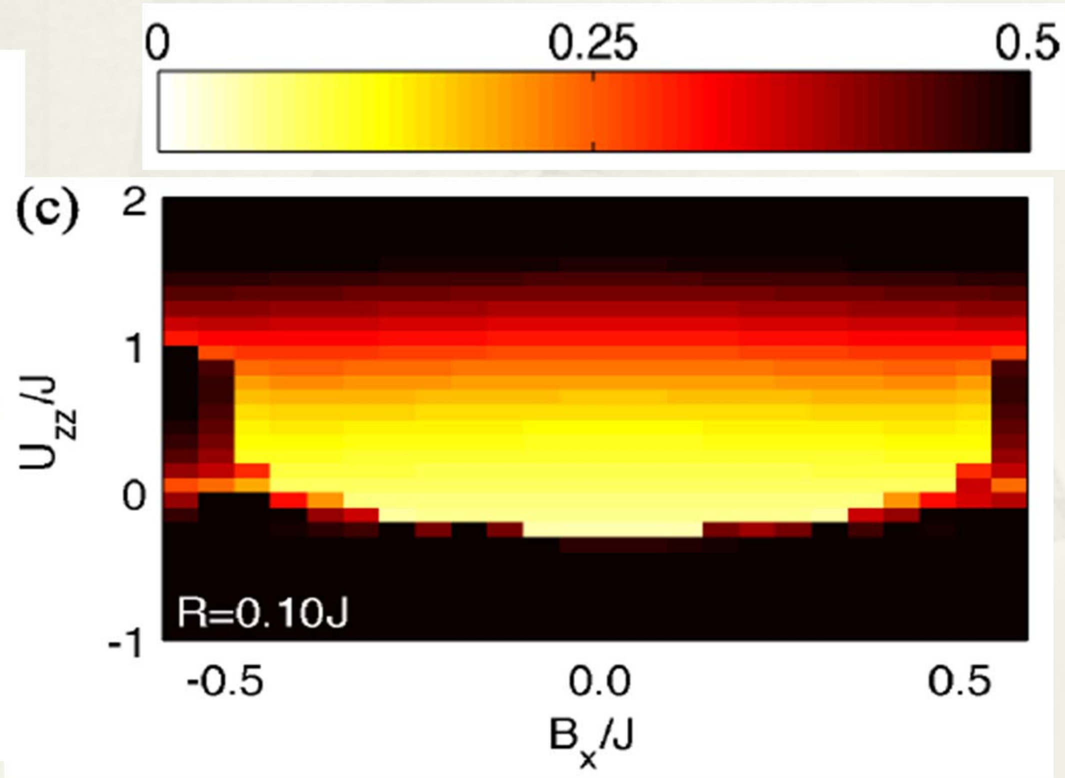
The case without symmetry protection

- Symmetry breaking perturbations

$$H_1 = R\sum_j [S^z_j(S^x_j S^x_{i+1} + S^y_j S^y_{j+1}) - S^z_{j+1}(S^x_j S^x_{j+1} + S^y_j S^y_{j+1}) + \mathrm{H.c.}]$$



Entanglement entropy



Entanglement spectrum

Pollmann,et.al. PRB 81, 064439 (2010)

Symmetry is crucial in 1D

- Symmetry protect the gapped nontrivial phases

- How to classify all the gapped symmetric nontrivial phases?

  Entanglement spectrum? not sufficient!

  Projective representations of the symmetry group.

Example

- *S=1* Model

$$H = \sum_i \left[\cos\theta S_{x,i} S_{x,i+1} + \sin\theta [\cos\phi (S_{y,i} S_{y,i+1} + S_{z,i} S_{z,i+1}) + \sin\phi (S_{xz,i} S_{xz,i+1} + S_{xy,i} S_{xy,i+1})] \right]$$

where $$S_{mn} = S_m S_n + S_n S_m, \qquad m, n = x, y, z$$

symmetry group: D_2+T

$$\theta = 0 \quad : \quad \text{Ising-like}$$

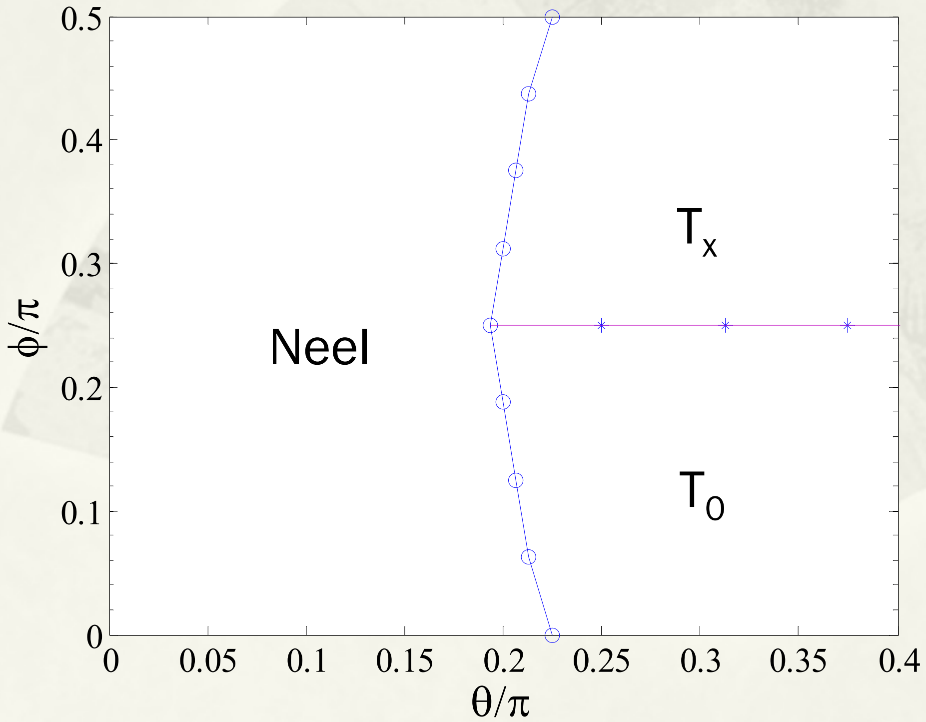
$$\theta = \frac{\pi}{2} \quad : \quad \text{XY-like}$$

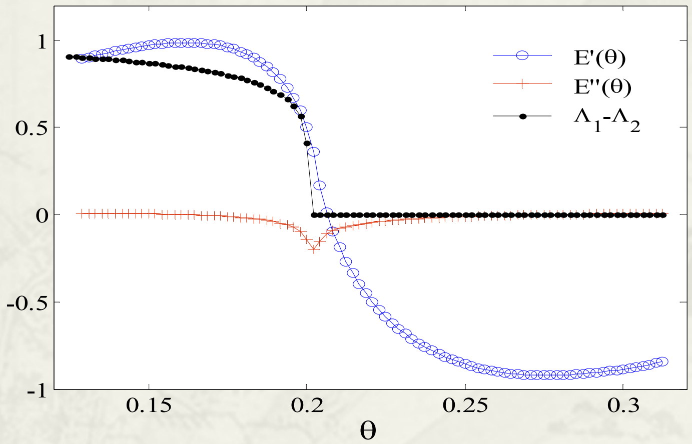
ϕ : rotation between S_y and S_{xz}, S_z and S_{xy}

Phase diagram

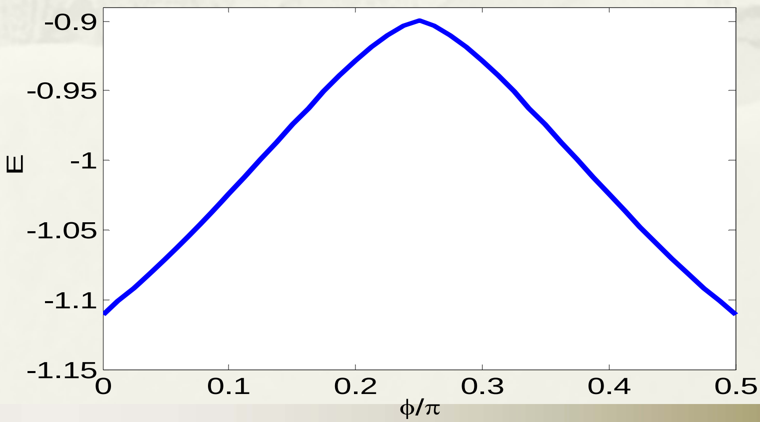
- Two nontrivial phases

$$H = \sum_i \left[\cos\theta S_{x,i}S_{x,i+1} + \sin\theta[\cos\phi(S_{y,i}S_{y,i+1}+S_{z,i}S_{z,i+1}) + \sin\phi(S_{xz,i}S_{xz,i+1} + S_{xy,i}S_{xy,i+1})]\right]$$





Φ= π/8, 3π/8



θ= 3π/8

Classification of SPT orders in 1D

- MPS form for gapped states in 1D

$$|\phi\rangle = \sum_{\{m_i\}} \mathrm{Tr}\left[\left(A_1^{m_1}|m_1\rangle\right)\left(A_2^{m_2}|m_2\rangle\right)\ldots\left(A_N^{m_N}|m_N\rangle\right)\right]$$

G. Vidal, Phys. Rev. Lett. 91, 147902 (2003)

- the MPS is invariant under symmetry group G

$$\hat{g}|\phi\rangle \propto |\phi\rangle$$

$$\hat{g}A^m = [u(g)]_{mm'}A^{m'} = e^{i\alpha(g)}M(g)^+ A^m M(g)$$

  - $M(g)$ projective representation (representation of end states)

Chen, Gu, Wen, Phys. Rev. B 83, 035107 (2011)

Projective representations of symmetry group

- Multiplication up to a phase factor

$$\sum_{m'} u(g_1)_{mm'} A^{m'} = e^{i\alpha(g_1)} M(g_1)^{+} A^{m} M(g_1)$$

$$\sum_{m'} u(g_2)_{mm'} A^{m'} = e^{i\alpha(g_2)} M(g_2)^{+} A^{m} M(g_2)$$

$$\sum_{m'} u(g_1 g_2)_{mm'} A^{m'} = e^{i\alpha(g_1 g_2)} M(g_1 g_2)^{+} A^{m} M(g_1 g_2)$$

- Linear representations

$$u(g_1)u(g_2) = u(g_1 g_2), \qquad e^{i\alpha(g_1)} e^{i\alpha(g_2)} = e^{i\alpha(g_1 g_2)}$$

Not important if the system has no translation symmetry

- Projective representation:

$$M(g_1)M(g_2) = M(g_1 g_2) e^{i\theta(g_1, g_2)}$$

Projective representations

- Equivalence class

$$\left.\begin{aligned} M(g)' &= M(g)e^{i\varphi(g)} \\ M(g_1)'M(g_2)' &= M(g_1g_2)'e^{i\theta'(g_1,g_2)} \end{aligned}\right\} \Rightarrow e^{i\theta'(g_1,g_2)} = e^{i\theta(g_1,g_2)} \frac{e^{i\varphi(g_1)}e^{i\varphi(g_1)}}{e^{i\varphi(g_1g_2)}}$$

  - $e^{i\theta'(g1,g2)}$ and $e^{i\theta(g1,g2)}$ belong to the same class
  - Each class $\omega \in H^2(G,U(1))$ ⇔ projective representation
- Projective representation ⇔ SPT phase
- Example: SO(3) has two classes of projective representations

$$SO(3) \begin{cases} \text{integer spin:} & \text{linear REP} \\ \text{half-integer spin:} & \text{nontrivial projective REP} \end{cases}$$

Classification of SPT orders with different symmetries

- on-site symmetry G ➔ ω
- on-site symmetry G + translational symmetry ➔ (α,ω)

Symmetry of Hamiltonian	Number of Different Phases
None	1
$SO(3)$	2
D_2	2
T	2
$SO(3)+T$	4
D_2+T	16
Trans. $+U(1)$	∞
Trans. $+SO(3)$	2
Trans. $+D_2$	$4\times 2=8$
Trans. $+P$	4
Trans. $+T$	2
Trans. $+P+T$	8
Trans. $+SO(3)+P$	8
Trans. $+D_2+P$	128
Trans. $+SO(3)+T$	4
Trans. $+D_2+T$	64
Trans. $+SO(3)+P+T$	16
Trans. $+D_2+P+T$	1024

Chen, Gu, Wen, Phys. Rev. B 83, 035107 (2011); arXiv:1103.3323

Questions

- Is the classification complete? YES!
  - Symmetry operators
  - Local unitary transformations
- Does the projective representations give complete information of all the SPT phases? YES!!

We will illustrate it by spin models with D_2+T symmetry

$S=1$ spin chains with D_2+T symmetry

- General Hamiltonian

$$
\begin{aligned}
H_{D_{2h}} = \sum_i [& a_1 S_{x,i}^2 S_{x,j}^2 + a_2 (S_{x,i}^2 S_{y,j}^2 + S_{y,i}^2 S_{x,j}^2) \\
& + a_3 S_{y,i}^2 S_{y,j}^2 + a_4 (S_{x,i}^2 S_{z,j}^2 + S_{z,i}^2 S_{x,j}^2) \\
& + a_5 S_{z,i}^2 S_{z,j}^2 + a_6 (S_{y,i}^2 S_{z,j}^2 + S_{z,i}^2 S_{y,j}^2) \\
& + b_1 S_{x,i} S_{x,j} + b_2 S_{yz,i} S_{yz,j} + c_1 S_{y,i} S_{y,j} \\
& + c_2 S_{xz,i} S_{xz,j} + d_1 S_{z,i} S_{z,j} + d_2 S_{xy,i} S_{xy,j} \\
& + e_1 S_{x,i}^2 + e_2 S_{y,i}^2 + e_3 S_{z,i}^2].
\end{aligned}
$$

  - Difficult to study directly
  - Symmetry breaking phases: well understood and boring
  - We will focus on symmetric (SPT) phases

General properties of SPT phases

- Physical degrees of freedom

linear representations of D_2+T

- Edge (or internal) degrees of freedom

projective representations of D_2+T

How to calculate the projective representations

- Central extension of G ➔ $R(G)$ $\quad\quad$ $G \approx R(G)/C$
- Linear Reps of $R(G)$ ⇔ projective Reps of G
- Example: projective representation of D_2 group

$D_2 = \{E, R_x, R_y, R_z\}$

$R(D_2) = \{E, P, Q, PQ, P^2, Q^2, P^2Q, P^3Q\}$

TABLE III: Multiplication table of $R_1(D_2)$. Notice that $P^4 = Q^4 = E$, $P^2 = Q^2$ and $QP = P^3Q$.

	E	P^2	P^3	P	Q	P^2Q	PQ	P^3Q
E	E	P^2	P^3	P	Q	P^2Q	PQ	P^3Q
P^2	P^2	E	P	P^3	P^2Q	Q	P^3Q	PQ
P^3	P^3	P	P^2	E	P^3Q	PQ	Q	P^2Q
P	P	P^3	E	P^2	PQ	P^3Q	P^2Q	Q
Q	Q	P^2Q	PQ	P^3Q	P^2	E	P	P^3
P^2Q	P^2Q	Q	P^3Q	PQ	E	P^2	P^3	P
PQ	PQ	P^3Q	P^2Q	Q	P^3	P	P^2	E
P^3Q	P^3Q	PQ	Q	P^2Q	P	P^3	E	P^2

projection:

E, P^2 ➔ E

P, P^3 ➔ R_z

Q, P^2Q ➔ R_x

PQ, P^3Q ➔ R_y

TABLE V: Projection from $R_1(D_2)$ to D_2

$R_1(D_2)$	E, P^2	P, P^3	Q, P^2Q	PQ, P^3Q
D_2	E	R_z	R_x	R_y
rotation π of $J = 1/2$ (up to a phase factor)	I	$i\sigma_z$	$i\sigma_x$	$i\sigma_y$

L. L. Boyle and Kerie F. Green, Mathematical and Physical Sciences, A 288, 237 (1978)

Combined symmetry: $D_{2h}=D_2+T$

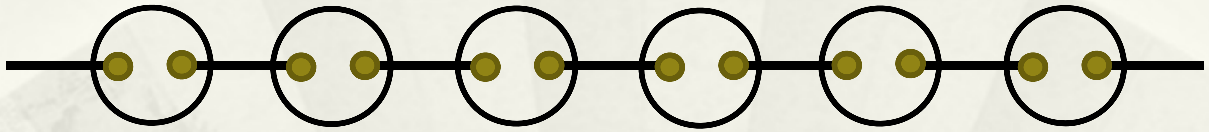
	R_z	R_x	T	...	ω,β,γ	dim.	active operators[a]	spin models ($S=1$)
E_0	1	1	K	...	1, 1,A	1		chain(trivial phase)
E_0'	I	I	$\sigma_y K$	...	1,-1,A	2	$(S_{xyz}, S_{xyz}, S_{xyz})$[b]	ladder
E_1	I	$i\sigma_z$	$\sigma_y K$	...	1,-1,B_1	2	(S_z, S_z, S_{xyz})	ladder
E_1'	I	$i\sigma_z$	$\sigma_x K$	...	1, 1,B_1	2	$(S_{xy}, S_{xy}, S_{xyz})$	ladder
E_3	σ_z	I	$i\sigma_y K$	...	1,-1,B_3	2	(S_x, S_x, S_{xyz})	ladder
E_3'	σ_z	I	$i\sigma_x K$	...	1, 1,B_3	2	$(S_{yz}, S_{yz}, S_{xyz})$	ladder
E_5	$i\sigma_z$	σ_x	IK	...	-1, 1,A	2	(S_{yz}, S_y, S_{xy})	chain(T_y phase)
E_5'	$I\otimes i\sigma_z$	$I\otimes\sigma_x$	$\sigma_y\otimes IK$	...	-1,-1,A	4	$(S^3_{xyz}, S^3_x, S^1_{yz}, S^3_{xz}, S^1_y, S^3_z, S^1_{xy})$[c]	ladder
E_7	σ_z	$i\sigma_z$	$i\sigma_x K$	...	1, 1,B_2	2	$(S_{xz}, S_{xz}, S_{xyz})$	ladder
E_7'	σ_z	$i\sigma_z$	$i\sigma_y K$	...	1,-1,B_2	2	(S_y, S_y, S_{xyz})	ladder
E_9	$i\sigma_z$	σ_x	$i\sigma_x K$	...	-1, 1,B_3	2	(S_{yz}, S_{xz}, S_z)	chain(T_z phase)
E_9'	$I\otimes i\sigma_z$	$I\otimes\sigma_x$	$\sigma_y\otimes i\sigma_x K$	...	-1,-1,B_3	4	$(S^3_{xyz}, S^3_x, S^1_{yz}, S^3_y, S^1_{xz}, S^3_{xy}, S^1_z)$[d]	ladder
E_{11}	$i\sigma_z$	$i\sigma_x$	$\sigma_z K$	...	-1, 1,B_1	2	(S_x, S_{xz}, S_{xy})	chain(T_x phase)
E_{11}'	$I\otimes i\sigma_z$	$I\otimes i\sigma_x$	$\sigma_y\otimes\sigma_z K$	...	-1,-1,B_1	4	$(S^3_{xyz}, S^3_{yz}, S^1_x, S^3_y, S^1_{xz}, S^3_z, S^1_{xy})$[e]	ladder
E_{13}	$i\sigma_z$	$i\sigma_x$	$i\sigma_y K$	...	-1,-1,B_2	2	(S_x, S_y, S_z)	chain(T_0 phase)
E_{13}'	$I\otimes i\sigma_z$	$I\otimes i\sigma_x$	$\sigma_y\otimes i\sigma_y K$	...	-1, 1,B_2	4	$(S^3_{xyz}, S^3_{yz}, S^1_x, S^3_{xz}, S^1_y, S^3_{xy}, S^1_z)$[f]	ladder

16 Projective representations: description of free-edge states.

arXiv:1101.5680; arXiv:1105.6021

Construct MPS with known edge states

- *S=1* AKLT model (Haldane phase)



$$A^m = B^T (C^m)^*$$

B and C^m are CG coefficients

- Parent Hamiltonian (sum of projectors)

$$H = \sum_i P_2(S_i + S_{i+1}) \propto \sum_i [S_i \cdot S_{i+1} + \frac{1}{3}(S_i \cdot S_{i+1})^2]$$

  - Simpler model in the same phase

$$H = \sum_i S_i \cdot S_{i+1}$$

$S=1$ spin chains with D_2+T symmetry

- SPT phases and their Hamiltonians
  - Trivial phase (ground state $\Psi=|\alpha, \alpha, \ldots, \alpha>$, $\alpha=x, y, z$)

$$H = \sum_i [S_i \cdot S_{i+1} + U(S_i^\alpha)^2]$$

  - T_0 phase (the usual Haldane phase)

$$H_0 = \sum_i J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z$$

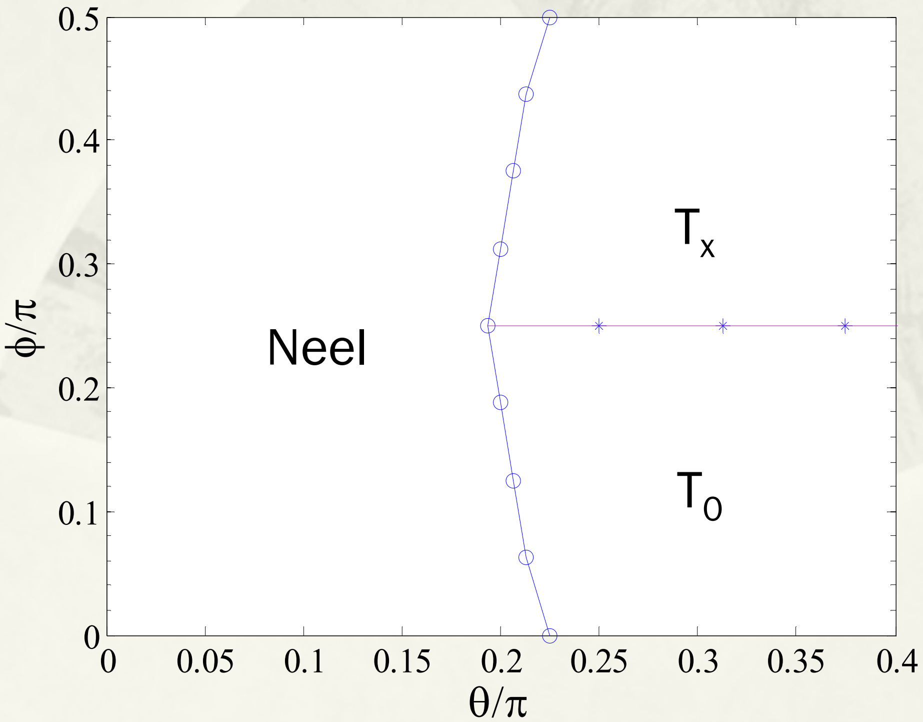
  - T_x phase $$H_x = \sum_i J_x S_i^x S_{i+1}^x + J_y S_i^{xz} S_{i+1}^{xz} + J_z S_i^{xy} S_{i+1}^{xy}$$
  - T_y phase $$H_y = \sum_i J_x S_i^{yz} S_{i+1}^{yz} + J_y S_i^y S_{i+1}^y + J_z S_i^{xy} S_{i+1}^{xy}$$
  - T_z phase $$H_z = \sum_i J_x S_i^{yz} S_{i+1}^{yz} + J_y S_i^{xz} S_{i+1}^{xz} + J_z S_i^z S_{i+1}^z$$

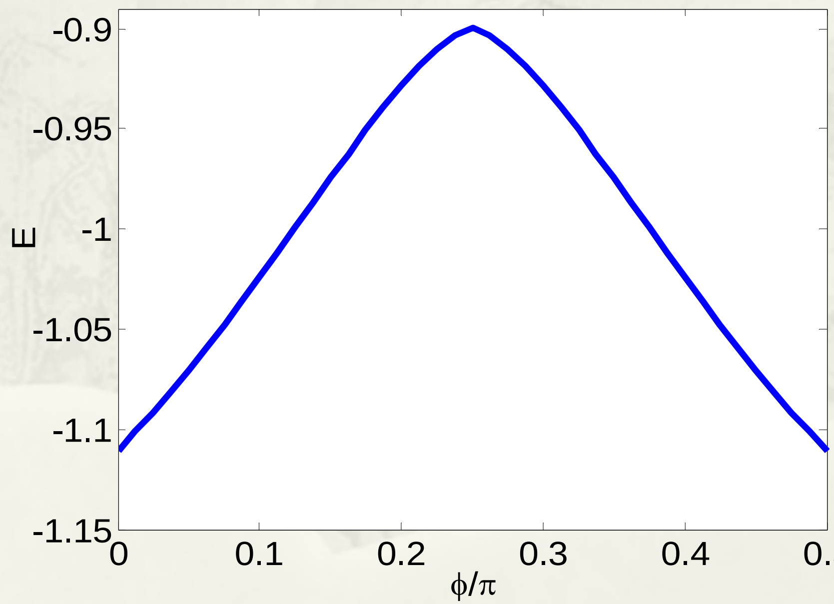
Phase transitions between different phases

- Model Hamiltonian

$$H = \sum_i \left[\cos\theta S_{x,i} S_{x,i+1} + \sin\theta [\cos\phi (S_{y,i} S_{y,i+1} + S_{z,i} S_{z,i+1}) + \sin\phi (S_{xz,i} S_{xz,i+1} + S_{xy,i} S_{xy,i+1})] \right]$$

- 1st order transition between different nontrivial SPT phases

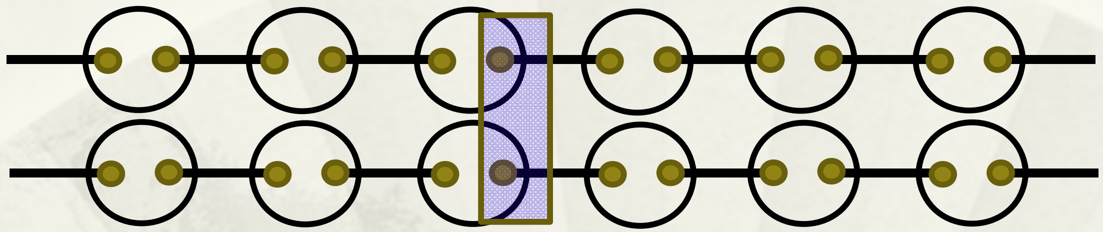




$\theta = 3\pi/8$

Realization of other phases in spin ladders

- Direct product of two spin chains



$$M_1 \otimes M_2 = M_3 \oplus M_3$$

Examples:

$$E_9 \otimes E_{13} = E_1 \oplus E_1$$

$$E_9 \otimes E_{11} = E_7 \oplus E_7$$

...

All the 16 phases are realized.

	R_z	R_x	T	...
E_0	1	1	K	...
E'_0	I	I	$\sigma_y K$	...
E_1	I	$i\sigma_z$	$\sigma_y K$	...
E'_1	I	$i\sigma_z$	$\sigma_x K$	...
E_3	σ_z	I	$i\sigma_y K$	...
E'_3	σ_z	I	$i\sigma_x K$	...
E_5	$i\sigma_z$	σ_x	IK	...
E'_5	$I \otimes i\sigma_z$	$I \otimes \sigma_x$	$\sigma_y \otimes IK$	...
E_7	σ_z	$i\sigma_z$	$i\sigma_x K$	...
E'_7	σ_z	$i\sigma_z$	$i\sigma_y K$	...
E_9	$i\sigma_z$	σ_x	$i\sigma_x K$	...
E'_9	$I \otimes i\sigma_z$	$I \otimes \sigma_x$	$\sigma_y \otimes i\sigma_x K$	...
E_{11}	$i\sigma_z$	$i\sigma_x$	$\sigma_z K$	...
E'_{11}	$I \otimes i\sigma_z$	$I \otimes i\sigma_x$	$\sigma_y \otimes \sigma_z K$	...
E_{13}	$i\sigma_z$	$i\sigma_x$	$i\sigma_y K$	...
E'_{13}	$I \otimes i\sigma_z$	$I \otimes i\sigma_x$	$\sigma_y \otimes i\sigma_y K$	...

Distinguish different phases from Edge states

- Small perturbations: magnetic field

$$H' = \sum_i \left(g_x B^x S_i^x + g_y B^y S_i^y + g_z B^z S_i^z \right)$$

- Bulk: gapped singlet, no response
- Edge: degenerate (like impurity spins)
  - Linear response (first order perturbation theory)
  - **Will the degeneracy be split by *H'*** ?

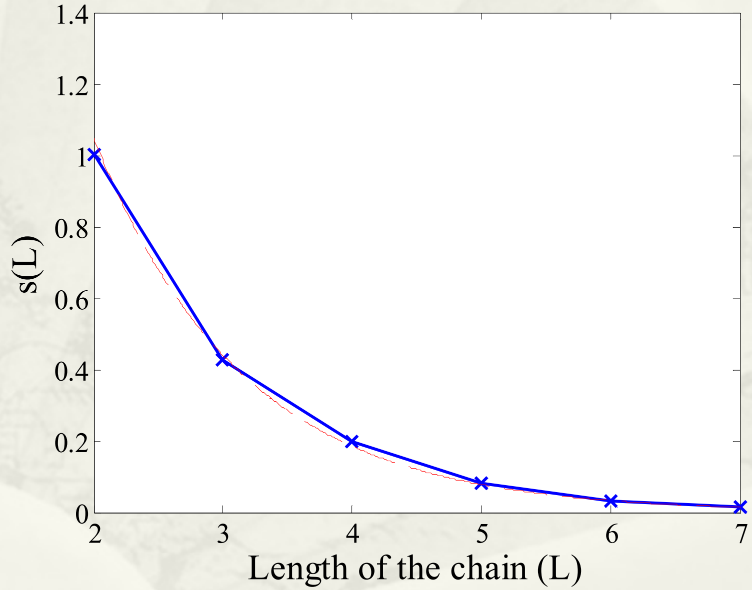
T_0 phase: Yes! B_x, B_y, B_z will polarize the end spins and split the ground state degeneracy.

T_x, T_y, T_z phases?

T_x phase

- T_x phase: **Will the magnetic field split the edge degeneracy?**
  - depends on direction of B
  - Numerical result (exact diagonalization):

$$<\sum S_i^x> \rightarrow 0,0,\pm 1,$$
$$<\sum S_i^y> \rightarrow 0,0,0,0,$$
$$<\sum S_i^z> \rightarrow 0,0,0,0.$$



In T_x phase $\left\langle \sum_i S_i^x \right\rangle = 0,0,\pm s$

with $\lim_{L\to\infty} s = 0$

T_y, T_z phases

- T_y phase

$$< \sum S_i^x > \rightarrow 0,0,0,0,$$
$$< \sum S_i^y > \rightarrow 0,0,\pm 1,$$
$$< \sum S_i^z > \rightarrow 0,0,0,0.$$

- T_z phase

$$< \sum S_i^x > \rightarrow 0,0,0,0,$$
$$< \sum S_i^y > \rightarrow 0,0,0,0,$$
$$< \sum S_i^z > \rightarrow 0,0,\pm 1.$$

Symmetry reason

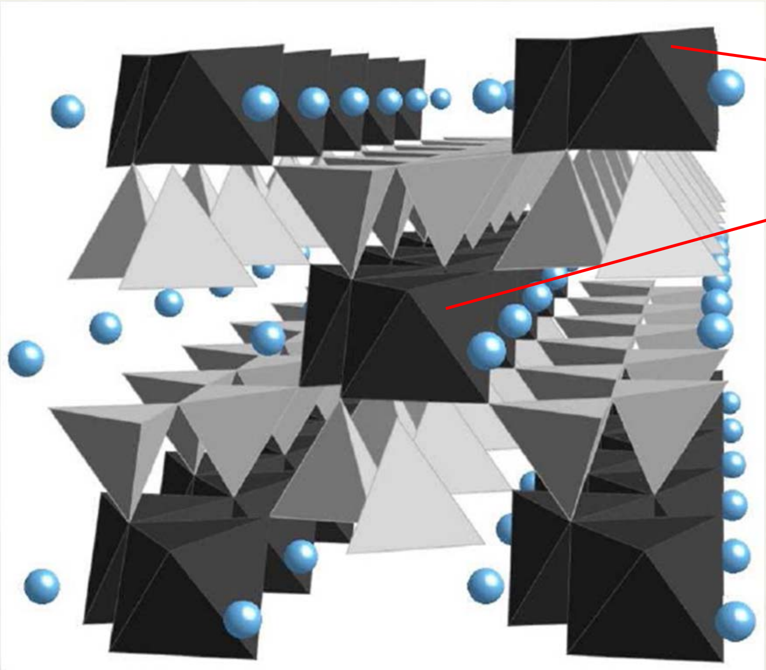
- Effective operators and physical perturbations
  - For 2 by 2 MPS (2-fold edge states), only three Pauli operators that split the degeneracy of the ground states
  - The Pauli operators form linear reps of the SG
  - The physical operators also form linear reps of the SG

$$M(g)^{-1}\sigma^{\alpha}M(g)=\eta_g(\sigma^{\alpha})\sigma^{\alpha}$$

$$u(g)^{-1}Ou(g)=\eta_g(O)O$$

  - $\eta(\sigma^{\alpha})=\eta(O)$ ➔ $\sigma^x \sim O$, they have the same matrix elements on the ground state subspace (up to a constant factor)
  - For example, in T_x phase: $\sigma^x \sim S^x, \quad \sigma^y \sim S^{xz}, \quad \sigma^z \sim S^{xy}$

Experimental measurements (I)



VO_6 V^{3+} S=1

Lattice structure of spin chain ($LiVSi_2O_6$)

edge states: magnetic impurities

- Curie's law: divergence of magnetic susceptibility at low temperatures

$$\chi_m(T) = \frac{N g_m^2 \mu_B^2}{3 k_B T}, \quad m = x, y, z$$

Experimental measurements (II)

- No edge states in the trivial phase

$$g_x,\ g_y,\ g_z \approx 0$$

- Nontrivial phases:
  - In T_0 phase: g_x, g_y, g_z are finite
  - In T_x phase: g_x is finite, g_y, $g_z \approx 0$
  - In T_y phase: g_y is finite, g_x, $g_z \approx 0$
  - In T_z phase: g_z is finite, g_x, $g_y \approx 0$

Other perturbations

- Perturbations by S^α are not sufficient to distinguish all the SPT phases. We need perturbations associated to the quadrupole operators $S^{\alpha\beta}$,

$$H' = \sum_i g_{xy} \left(\nabla_y B_x + \nabla_x B_y \right)_i S_i^{xy} + \ldots$$

- From the different responses of the edge states to perturbations of B_x, B_y, B_z, and $\frac{\partial B^x}{\partial y}$, … all the phases can be distinguished.

Recent progress

- Generalization to higher dimensions
  - 1D: $H^2(G, U(1))$
  - 2D: $H^3(G, U(1))$
  - 3D: $H^4(G, U(1))$
- Reference:

2D toy model realizing nontrivial $H^3(Z_2, U(1))$

arXiv:1106.4752

Classifying dD SPT phase through topological nonlinear sigma model

arXiv:1106.4772

Conclusion

- There are 16 different SPT phases in 1D protected by D_2+T symmetry
- They can be realized with *S=1* spin chains and spin ladders
- All the SPT phases are completely characterized by their edge spins (projective representations) and are experimentally distinguishable.

	R_z	R_x	T	...	ω, β, γ	dim.	active operators[a]	spin models ($S=1$)
E_0	1	1	K	...	1, 1,A	1		chain(trivial phase)
E'_0	I	I	$\sigma_y K$	...	1,-1,A	2	$(S_{xyz}, S_{xyz}, S_{xyz})$[b]	ladder
E_1	I	$i\sigma_z$	$\sigma_y K$	...	1,-1,B_1	2	(S_z, S_z, S_{xyz})	ladder
E'_1	I	$i\sigma_z$	$\sigma_x K$	...	1, 1,B_1	2	$(S_{xy}, S_{xy}, S_{xyz})$	ladder
E_3	σ_z	I	$i\sigma_y K$	...	1,-1,B_3	2	(S_x, S_x, S_{xyz})	ladder
E'_3	σ_z	I	$i\sigma_x K$	...	1, 1,B_3	2	$(S_{yz}, S_{yz}, S_{xyz})$	ladder
E_5	$i\sigma_z$	σ_x	IK	...	-1, 1,A	2	(S_{yz}, S_y, S_{xy})	chain(T_y phase)
E'_5	$I \otimes i\sigma_z$	$I \otimes \sigma_x$	$\sigma_y \otimes IK$	...	-1,-1,A	4	$(S^3_{xyz}, S^3_x, S^1_{yz}, S^3_{xz}, S^1_y, S^3_z, S^1_{xy})$[c]	ladder
E_7	σ_z	$i\sigma_z$	$i\sigma_x K$	...	1, 1,B_2	2	$(S_{xz}, S_{xz}, S_{xyz})$	ladder
E'_7	σ_z	$i\sigma_z$	$i\sigma_y K$	...	1,-1,B_2	2	(S_y, S_y, S_{xyz})	ladder
E_9	$i\sigma_z$	σ_x	$i\sigma_x K$	...	-1, 1,B_3	2	(S_{yz}, S_{xz}, S_z)	chain(T_z phase)
E'_9	$I \otimes i\sigma_z$	$I \otimes \sigma_x$	$\sigma_y \otimes i\sigma_x K$	...	-1,-1,B_3	4	$(S^3_{xyz}, S^3_x, S^1_{yz}, S^3_y, S^1_{xz}, S^3_{xy}, S^1_z)$[d]	ladder
E_{11}	$i\sigma_z$	$i\sigma_x$	$\sigma_z K$	...	-1, 1,B_1	2	(S_x, S_{xz}, S_{xy})	chain(T_x phase)
E'_{11}	$I \otimes i\sigma_z$	$I \otimes i\sigma_x$	$\sigma_y \otimes \sigma_z K$	...	-1,-1,B_1	4	$(S^3_{xyz}, S^3_{yz}, S^1_x, S^3_y, S^1_{xz}, S^3_z, S^1_{xy})$[e]	ladder
E_{13}	$i\sigma_z$	$i\sigma_x$	$i\sigma_y K$	...	-1,-1,B_2	2	(S_x, S_y, S_z)	chain(T_0 phase)
E'_{13}	$I \otimes i\sigma_z$	$I \otimes i\sigma_x$	$\sigma_y \otimes i\sigma_y K$	...	-1, 1,B_2	4	$(S^3_{xyz}, S^3_{yz}, S^1_x, S^3_{xz}, S^1_y, S^3_{xy}, S^1_z)$[f]	ladder

Thanks for attention!