Diffraction by an optical fractal grating

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We report experimental and theoretical studies of Fraunhofer diffraction pattern of a 15-level H-fractal grating. The diffraction pattern was found to exhibit self-similarity. In particular, the diffracted light was found to be more intense at higher spatial frequencies than at lower frequencies, in stark contrast to the diffraction patterns of wire gratings and grid gratings. Using Fourier transform theory, we show that this unusual behavior comes from the structural coherence of the H-fractal, which makes it favorable to use higher-order diffraction spectra for larger dispersion. In addition, the fractal dimension of the grating is shown analytically and experimentally to be two. © *2004 American Institute of Physics*. [DOI: 10.1063/1.1840112]

Fractals, such as Cantor sets and Koch curves, have unique self-similarity or scale invariance properties. Another surprising property of fractals is that their dimensions can take on fractional values. It has been proved that the similarity dimensions of the Cantor set and the triadic Koch curve are 0.631 and 1.262, respectively.[1] Due to these peculiar features, fractal structures are expected to have vastly different properties from structures with regular periodicity. In the regime of optics, it has been found that the diffraction patterns of fractal objects also exhibit self-similarity and the patterns themselves can be employed to determine the fractal dimensions of the objects.[2–6] Very recently, a metallic H-fractal plate has been shown to possess multiple pass bands and stop bands for electromagnetic waves over an ultrawide microwave frequency range and break down the rigid relationship between the band-gap frequencies and the periodicity of usual photonic band-gap structures.[7] Enhanced infrared transmission has also been observed for conducting films patterned with an inverted H-fractal structure.[8] It must be noted that fractals are generated from an infinite process in a mathematical sense, while they are defined by finite levels for physical applications. In this work, we report experimental and theoretical studies of the Fraunhofer diffraction pattern of a multilevel H-fractal grating in the visible regime. We found that the pattern, especially the light flux distribution, is very different from those obtained from ordinary gratings. We determined the fractal dimension of the grating from the diffraction pattern, which is consistent with our theoretical calculations.

The H-fractal was generated from a line of length a, defined as the first level of the structure and placed along the x axis of an x–y plane. The $(k+1)$th level structure contains 2^k lines, with the midpoint of each line perpendicularly connected to the ends of the kth level lines. The length of the $(k+1)$th level lines was scaled from that of the kth level line by a factor of 2(1), if k is an even (odd) number. With an increasing number of levels, the structure eventually approaches a space-filling curve that tiles a $2a\times 2a$ square.[7,8] In the following mathematical proof, the fractal dimension is shown analytically to be two, the same as the Euclidean dimension of the space that supports the fractal. For an H-fractal, the kth level structure contains 2^{k-1} lines, each being $1/2^{(k-1)/2}$ in length for odd k or $1/2^{(k/2)^{-1}}$ for even k, where the first-level line length has been assumed to be a unit. When the scale reduces down to $s=1/2^{n-1}$ (n being a natural number), the length of the H-fractal is measured as

$$\begin{aligned} u &= 1\times 1+1\times 2+\frac{1}{2}\times 4+\frac{1}{2}\times 8+\frac{1}{4}\times 16+\frac{1}{4}\times 32 \\ &\quad +\cdots+\frac{1}{2^{n-1}}\times 2^{2n-2}+\frac{1}{2^{n-1}}\times 2^{2n-1} \\ &= \sum_{k=1}^{n}\left(\frac{1}{2^{k-1}}\times 2^{2k-2}+\frac{1}{2^{k-1}}\times 2^{2k-1}\right) \\ &= 3\times\sum_{k=1}^{n}2^{k-1}=3\times(2^n-1). \end{aligned}$$

Following the definition of the dimension of a fractal, $u=s^{-(D-1)}$, the dimension of the H-fractal is

$$\begin{aligned} D &= 1+\lim_{n\to\infty}\left(\frac{\log u}{\log 1/s}\right)=1+\lim_{n\to\infty}\left[\frac{\log 3(2^n-1)}{\log 2^{n-1}}\right] \\ &= 1+1=2. \end{aligned}$$

Figure 1(a) shows a snapshot of part of our sample, a 15-level H-fractal grating. The length a of the first-level line is 147.7 μm. The width w of the chrome line is 0.7 μm. The opaque fractal pattern was printed on a very thin quartz substrate via photolithography. A frequency-doubled YAG laser beam of wavelength 532 nm (green) and a He–Ne laser beam of wavelength 633 nm (red) were combined collinearly and incident normally on the fractal grating. A lens with a 200 mm focal length was placed behind the grating. The distribution of the light field in the focal plane was recorded using a digital camera (Canon Power Shot S40), as shown in Fig. 1(b). It is seen clearly that the diffraction pattern is self-similar with a 1/2 scaling factor, particularly near the optical

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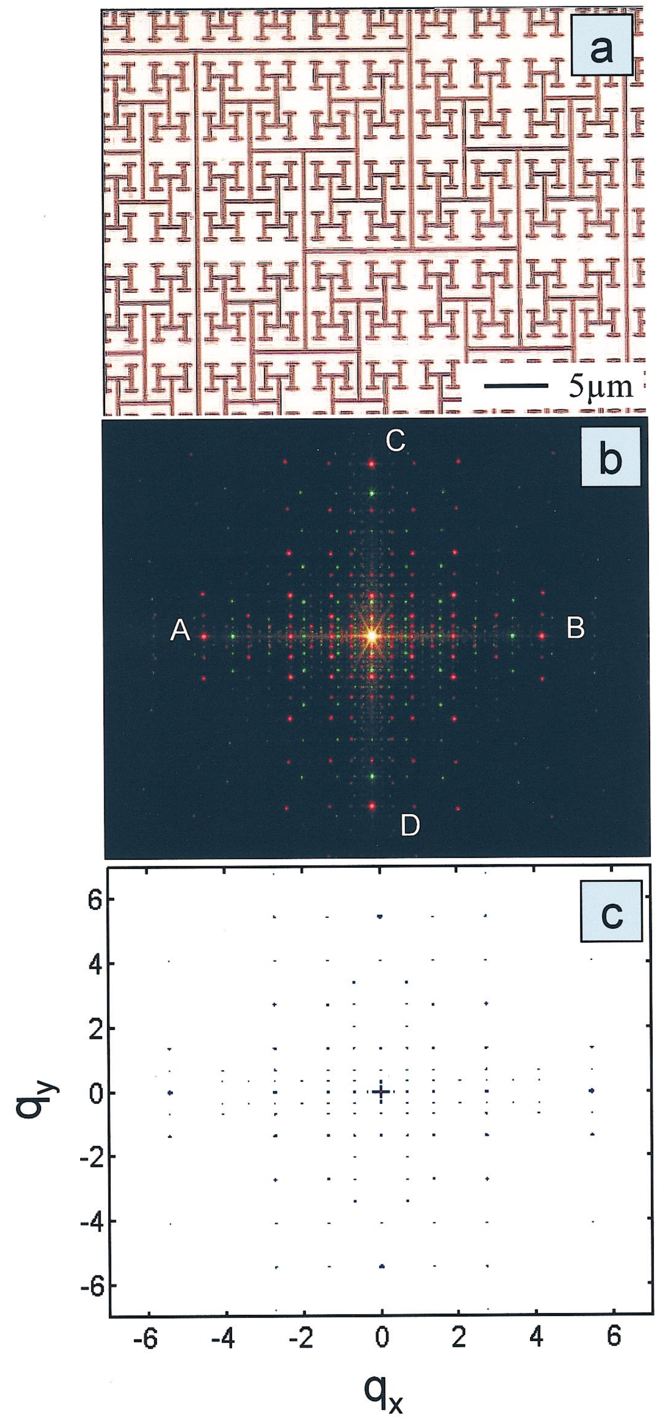

FIG. 1. (Color) (a) A part of the 15-level H-fractal grating, (b) the experimental Fraunhofer diffraction pattern under two illuminating wavelengths, 532 nm (green) and 633 nm (red), (c) the calculated diffraction spectrum of the fractal grating.

center. The intensity distribution is different in x and y direction, which is caused by anisotropy of the H-fractal of finite levels.

For our Fraunhofer diffraction experiment arrangement, the resultant diffraction pattern corresponds to the exact Fourier transform of the grating. We analytically calculated the diffraction pattern of the 15-level H-fractal grating using Fourier transformation, shown in Fig. 1(c). It is in good agreement with the experimental observations. In the calculation, we decomposed the H-fractal into various levels, because every level structure is actually a grid grating whose

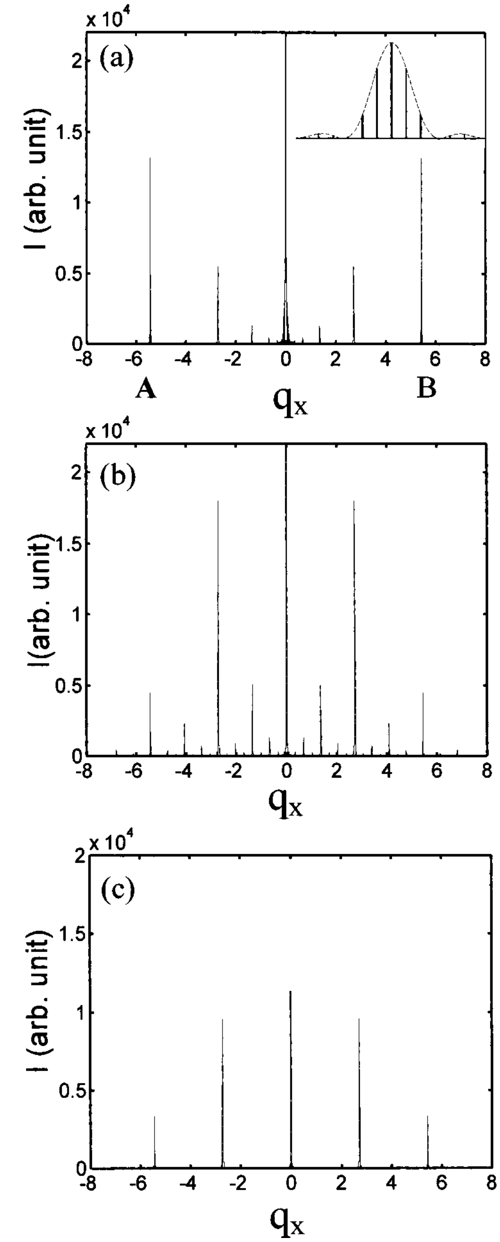

FIG. 2. (a) The intensity distribution of the diffraction pattern along the q_x axis. The inset is the diffraction pattern of a wire grating. (b) The incoherent superposition of the diffracted light from different levels of the H-fractal. (c) The 90°-rotated diffraction pattern of the 15th level structure. In the plots, the quantity I is unnormalized, representing the absolute intensity.

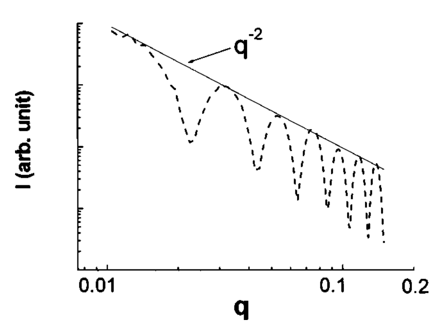

FIG. 3. The log–log plot of the diffracted light intensity vs the frequency modulus (dot line). The solid line is the relation, $I=q^{-2}$.

Fourier spectrum is treated in many standard optics textbooks.[9] The decomposition helped us to distinguish the contribution of each level to the diffraction pattern. The Fourier transform of the kth level structure is expressed as

$$G_k(q_x,q_y)=\begin{cases}\dfrac{1}{2^{n-1}}\mathrm{sinc}\left(q_x\dfrac{a}{2^n}\right)\mathrm{sinc}\left(q_y\dfrac{w}{2}\right)\dfrac{\sin(q_xa)}{\sin(q_xa/2^{n-1})}\dfrac{\sin(q_ya)}{\sin(q_ya/2^{n-1})}, & k=2n-1\\[2ex] \dfrac{1}{2^{n-1}}\mathrm{sinc}\left(q_x\dfrac{w}{2}\right)\mathrm{sinc}\left(q_y\dfrac{a}{2^n}\right)\dfrac{\sin(q_xa)}{\sin(q_xa/2^{n})}\dfrac{\sin(q_ya)}{\sin(q_ya/2^{n-1})}, & k=2n,\end{cases}$$

where q_x and q_y are the spatial frequency components and n is a natural number. So the amplitude of the diffracted field of the entire grating is proportional to the summation over all levels as $\Sigma_{k=1}^{15}G_k(q_x,q_y)$.

One important difference between H-fractal grating and most conventional gratings is that the diffraction pattern is unvaried under scaling and not periodic. Another difference is that the diffracted spots become brighter when moving away from the center, which is in stark contrast to those obtained from ordinary transmission gratings, such as grid gratings. It can be seen in Fig. 1(b) that there are four strongly bright spots marked as A, B, C, and D symmetrically located along the q_x and q_y axes. To make this point clearer, we plot in Fig. 2(a) the calculated intensity distribution along the q_x axis, which shows that the scaling property still holds for large diffraction angles. It is obvious from Fig. 2(a) that the peaks become more intense with increasing modulus, $|q_x|$, and are invariant over a scaling of a factor of two at the same time. For contrast, we show schematically the diffraction pattern of a wire grating in the inset of Fig. 2(a), where the peak intensity decreases as the diffraction angle becomes larger.

In Fig. 2(b), we plot the intensity map of the summation $\Sigma_{k=1}^{15}|G_k(q_x,q_y)|^2$ along the same axis. Comparing this with Fig. 2(a), we see that it is the interference term $G_i(\mathbf{q})G_j(\mathbf{q})(i\neq j)$ in the squared summation $|\Sigma_{k=1}^{15}G_k(q_x,q_y)|^2$ that shifts some light flux out of low spatial frequencies into higher frequencies and smears out many periodic tiny peaks to form the clean and self-similar distribution as in Fig. 2(a). Thus, the anomalous distribution of the diffracted light flux of the H-fractal grating originates from coherent superposition of the light diffracted from different levels and is determined only by the intrinsic characteristics of the grating. This conclusion can be confirmed by the fact that the diffraction patterns obtained at two wavelengths scale as the wavelength [see Fig. 1(b)]. It is worth noting that, for spectroscopic purposes, this diffracted light intensity distribution is desirable because of enhanced intensity of diffraction orders at large diffraction angle, which can lead to improved dispersion capacity. This is shown by comparing the fractal grating with its 15th level structure which is a grid grating with the same period, $a/64$, along two directions [see Fig. 2(c)]. The diffraction pattern of the grid grating has been rotated with respect to the optical center by 90° to facilitate the comparison. It is found that the light flux at $q_x=\pm5.5$ where the second-order spectra reside in Fig. 2(c) is enhanced and develops into diffraction peaks A and B after forming the complete fractal. The angular dispersion at the four on-axis diffraction spots, A, B, C, and D, was measured to be 0.26′/Å, which is compatible with the angular dispersion of a transmission grating of 800 lines/mm operating in the first-order spectrum and at the wavelength of 532 nm.

The fractal dimension of an object is included in the diffracted field according to the power law, $I(q)\propto q^{-D}$, where I is the diffracted field intensity, $q=\sqrt{q_x^2+q_y^2}$ is the modulus of the spatial frequency, and D is the fractal dimension. The relation holds only in a limited region of $1/L\ll q\ll 1/l$, which is defined in terms of the lower and upper cutoffs, L and l, of the scaling range of the object.[2] For the H-fractal, L and l are characterized by the longest line and the shortest line, respectively. We calculated the intensity variation of the diffracted light with respect to the modulus, q, in the limited region, as shown in Fig. 3. Despite many dips appearing in the curve owing to the regularity of the grating, its trend and envelope comply with the power law, q^{-2}. So the fractal dimension is $D=2$, which means that the H-fractal is not merely a curve, but a curve that actually occupies some area on a plane.

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