A Proposal for the Worldvolume Action of Multiple M5-Branes

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based on


Outline

1. Introduction

2. Non-abelian action for multiple M5-branes
   - Perry-Schwarz action for a single M5-brane

3. Non-abelian self-dual string solution

4. Discussions
1 Introduction

2 Non-abelian action for multiple M5-branes
   • Perry-Schwarz action for a single M5-brane

3 Non-abelian self-dual string solution

4 Discussions
Mysteries of M5-branes

What we know:

- The low energy worldvolume dynamics is given by a 6d (2,0) SCFT with $SO(5)$ R-symmetry.  
  \[ \text{(Strominger, Witten)} \]
  
The (2,0) tensor multiples contains 5 scalars and a selfdual antisymmetric 3-form field strength + fermions.  
  \[ \text{(Gibbons, Townsend; Strominger; Kaplan, Michelson)} \]

What we don’t know:

- What is the form of the gauge symmetry for multiple M5-branes?
- **Interacting self-dual dynamics** on M5-branes worldvolume?
Self-dual dynamics for multiple M5-branes (?)

- Generally, it is well known to be difficult to write down a Lorentz invariant action for self-dual dynamics.  
  (Siegel 84; Floreanini, Jackiw 87)

- For a single M5 case, problem solved by sacrificing manifest 6d Lorentz symmetry.  
  (Henneaux-Teitelboim 88; Perry-Schwarz 97)

  The action was later generalized to include kappa symmetry  
  (Aganagic, Park, Popescu, Schwarz, 97)

  Covariant construction was given by PST  
  (Pasti-Sorokin-Tonin)

- Not clear how to do this for $N > 1$ due to the other problem that an appropriate generalization of the tensor gauge symmetry was not known.
Enhanced gauge symmetry of multiple M5-branes (?)

- For multiple D-branes, symmetry is enhanced from $U(1)$ to $U(N)$:
  \[
  \delta A^a_\mu = \partial_\mu \Lambda^a + [A_\mu, \Lambda]^a, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + [A_\mu, A_\nu]^a.
  \]

- For multiple M5-branes, it is not known how to non-Abelianize 2-form (or higher form) gauge fields:
  \[
  \delta B^a_{\mu\nu} = \partial_\mu \Lambda^a_\nu - \partial_\nu \Lambda^a_\mu + (?), \quad H^a_{\mu\nu\lambda} = \partial_\mu B^a_{\nu\lambda} + \partial_\nu B^a_{\lambda\mu} + \partial_\lambda B^a_{\mu\nu} + (?).
  \]
  to have nontrivial self interaction.

- Moreover, exists no-go theorems: there is no nontrivial deformation of the Abelian 2-form gauge theory if *locality* of the action and the transformation laws are assumed.
  \[(\text{Henneaux; Bekaert; Sevrin; Nepomechie})\]

- These no-go theorems, however, suggest an important direction of given up locality.
The need of nonlocality for M5-branes should not be surprising: ABJM and BLG theory for multiple M2-branes are also non-local.

We will get around the no-go theorem by similarly introducing a set of auxiliary fields: the theory become nonlocal when these fields are eliminated using their equations of motion.
In this talk, I will explain a 6d proposal of the low energy worldvolume theory (Ko and Chu).

There is also a 5d proposal of the (2,0) theory as strong coupling limit of 5d SYM. (Douglas; Lambert, Papageorgakis and Schmidt-Sommerfeld)

analogy:
low energy SUGRA $\leftarrow$ quantum mechanical M-theory (proposed to be defined by D0s’ SQM)
low energy theory of M5-branes $\leftarrow$ quantum M5-branes theory (proposed to be defined by 5d SYM)

Other related works/approaches: Smith, Ho, Huang, Masuo, Samtleben, Sezgin, Wimmer, Wulff, Saemann, Wolf, Czech, Huang, Rozali, Tachikawa, H.C. Kim, S. Kim, Koh, K. Lee, S. Lee, Bolognesi, Maxfield, Sethi, Bak, Gustavsson, Hosomichi, Seong, Nosaka, Terashima, Kallen, Minahan, Nedelin, Zabzine, Palmer, Sorokin, Bandos, ...
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Review of the Perry-Schwarz formulation

- Perry-Schwarz formulation:
  1. A direction, say $x_5$, is treated differently: denote the 5d and 6d coord. by $x^\mu$ and $x^M = (x^\mu, x^5)$.
  2. The tensor gauge field potential is represented by a $5 \times 5$ antisymmetric tensor field $B_{\mu\nu}$. It can be thought of as a (tensor) gauge fixed formulation in which $B_{\mu 5}$ never appear.

- Perry and Schwarz considered the action:

$$S_0(B) = \frac{1}{2} \int d^6 x \left( -\tilde{H}_{\mu\nu} \tilde{H}^{\mu\nu} + \tilde{H}^{\mu\nu} \partial_5 B_{\mu\nu} \right)$$

where

$$\tilde{H}_{\mu\nu} := \frac{1}{6} \epsilon_{\mu\nu\rho\lambda\sigma} H_{\rho\lambda\sigma}.$$ 

Note that manifest Lorentz symmetry is lost.
EOM:

\[ \epsilon^{\mu\nu\rho\lambda\sigma} \partial_\rho (\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma}) = 0 \]

has the general solution

\[ \tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu, \quad \text{for arbitrary } \alpha_\mu. \]

The action is invariant under the gauge symmetry

\[ \delta B_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, \quad \text{for arbitrary } \phi_\mu. \]

This allows one to reduce the general solution to the EOM to the first order form

\[ \tilde{H}_{\mu\nu} = \partial_5 B_{\mu\nu}. \]

This is the self-duality equation in this theory.
Modified Lorentz symmetry

- The action has manifest 5d Lorentz invariance and a non-manifest Lorentz symmetry mixing $\mu$ with the 5 direction.

### Lorentz transformation (active view)

**Standard Lorentz transformation**

$$
\delta B_{\mu \nu} = (\Lambda \cdot L) B_{\mu \nu} + \delta_{\text{spin}} B_{\mu \nu}
$$

has an orbital part:

$$
\Lambda \cdot L = (\Lambda \cdot x) \partial_5 - x_5 (\Lambda \cdot \partial)
$$

and a spin part:

$$
\delta_{\text{spin}} B_{\mu \nu} = \Lambda_\nu B_{\mu 5} - \Lambda_\mu B_{\nu 5}.
$$

- PS proposed to consider the following transformation

$$
\delta B_{\mu \nu} = (\Lambda \cdot x) \tilde{H}_{\mu \nu} - x_5 (\Lambda \cdot \partial) B_{\mu \nu},
$$

where $\Lambda_\mu = \Lambda_5 \mu$ denote the corresponding infinitesimal transformation parameters.
1. On shell, it is equal to the standard Lorentz transformation

\[ \delta B_{\mu\nu} = (\Lambda \cdot x) \tilde{H}_{\mu\nu} - x_5 (\Lambda \cdot \partial) B_{\mu\nu} = (\Lambda \cdot x) \partial_5 B_{\mu\nu} - x_5 (\Lambda \cdot \partial) B_{\mu\nu} \]

2. Commutator:

\[ [\delta_{\Lambda_1}, \delta_{\Lambda_2}] B_{\mu\nu} = \delta_{\Lambda_{\alpha\beta}}^{(5d)} B_{\mu\nu} + \text{EOM} + \text{gauge symmetry} \]

where

\[ \delta B_{\mu\nu} = \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu, \quad \varphi_\nu = x^\alpha \Lambda_{\alpha\lambda} B_\nu^\lambda \]

is a gauge symmetry of the PS theory. So the PS transformation can be considered as a (modified) Lorentz transformation.

Note that modified Lorentz symmetry is typical of action of self-dual dynamics

(Siegel 84)
Non-abelian action

The idea is to try to generalize the Perry-Schwarz approach:

1. Represent the self-dual tensor gauge field by a $5 \times 5$ antisymmetric field $B^a_{\mu \nu}$ in the adjoint and giving up manifest 6d Lorentz symmetry.

2. Moreover we’ll introduce a set of YM gauge fields $A^a_\mu$ for gauge group $G$.

Comments

1. Motivation for 2:
   - Multiple M2-branes (ABJM) was used to probe the M5-branes system. The gauge non-invariance of the boundary Chern-Simons action was shown to imply the existence of a Kac-Moody current algebra on the worldsheet of multiple self-dual strings.
   - This was argued to induces a $U(N) \times U(N)$ gauge symmetry in the theory of $N$ coincident M5-branes. (Chu, Smith; Chu)

2. YM gauge field $A^a_\mu$ was also introduced in many other approaches, e.g. Ho, Huang, Masuo; Samtleben, Sezgin, Wimmer, Wulff; Sorokin’s talk; ... The main difference in our approach is that they are not propagating and so there is no extra degrees of freedom.
The Action

Our proposed action \( S = S_0 + S_E \) consists of two pieces:

\[
S_0 = \frac{1}{2} \int d^6 x \, \text{tr} \left( -\tilde{H}^{\mu\nu} \tilde{H}_{\mu\nu} + \tilde{H}^{\mu\nu} \partial_5 B_{\mu\nu} \right),
\]

is the non-abelian generalization of the Perry-Schwarz action, where

\[
H_{\mu\nu\lambda} = D_{[\mu} B_{\nu\lambda]}, \quad D_{\mu} = \partial_{\mu} + A_{\mu}.
\]

\[
S_E = \int d^5 x \, \text{tr} \left( (F_{\mu\nu} - c \int dx_5 \tilde{H}_{\mu\nu}) E^{\mu\nu} \right).
\]

where \( E_{\mu\nu} \) is a 5d auxiliary field implementing the constraint

\[
F_{\mu\nu} = c \int dx_5 \tilde{H}_{\mu\nu}
\]

- Note that there is no \( B_{\mu 5} \) and \( A_5 \). \( A_{\mu} \) and \( E_{\mu\nu} \) live in 5-dimensions
- Note also the presence of fields \( A_{\mu}, E_{\mu\nu} \) that is not expected from (2,0) susy.
The action is invariant under:

1. Yang-Mills gauge symmetry

\[ \delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda], \quad \delta B_{\mu\nu} = [B_{\mu\nu}, \Lambda], \quad \delta E_{\mu\nu} = [E_{\mu\nu}, \Lambda]. \]

2. Tensor gauge symmetry:

\[ \delta_T A_\mu = 0, \quad \delta_T B_{\mu\nu} = D_{[\mu\nu]}, \quad \delta_T E_{\mu\nu} = 0, \]

for arbitrary \( \Lambda_\mu(x^M) \) such that \([F_{\mu\nu}, \Lambda_\lambda] = 0\).

3. Moreover there is a gauge symmetry

\[ \delta E_{\mu\nu} = \alpha_{\mu\nu} \]

for arbitrary \( \alpha(x^\lambda) \) such that \([D_{[\mu\nu}, \Lambda_\lambda] = 0, \quad D^{\mu}\alpha_{\mu\lambda} = 0. \]
Property 1: Self-Duality

- EOM of $E_{\mu\nu}$ gives the constraint

$$F_{\mu\nu} = c \int dx_5 \tilde{H}_{\mu\nu}.$$ 

- EOM of $B_{\mu\nu}$

$$\epsilon^{\mu\nu\rho\lambda\sigma} D_\rho (\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma}) = 0$$

has the general solution

$$\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma} = \Phi_{\lambda\sigma},$$

where $D_{[\lambda \Phi_{\mu\nu}]} = 0$.

- Again with an appropriate fixing of the tensor gauge symmetry, one can reduce the second order EOM to the self-duality equation

$$\tilde{H}_{\mu\nu} = \partial_5 B_{\mu\nu}.$$
Property 2: Degrees of freedom

- After eliminating the aux field $E_{\mu\nu}$ and substituting the $F = H$ constraint, the resulting action becomes highly nonlinear and interacting. To count the degrees of freedom, we use the linearized theory.

  At the quadratic level, the non-abelian action is simply given by $\dim G$ copies of the Perry-Schwarz action. We obtain $3 \times \dim G$ degrees of freedom in $B_{\mu\nu}$.

- Our theory contains $3 \times \dim G$ degrees of freedom as required by (2,0) supersymmetry.
Property 3: Lorentz Symmetry

- The action is invariant under the 5-$\mu$ Lorentz transformation:

$$\delta B_{\mu\nu} = (\Lambda \cdot x) \tilde{H}_{\mu\nu} + x_5 \Lambda^\kappa H_{\kappa\mu\nu} + \Lambda^\kappa \phi_{\mu\nu\kappa},$$

where

$$\phi^a_{\mu\nu\kappa} = \int dy \ G^{ab\mu'\nu'}_{\mu\nu}(x, y) J^b_{\mu'\nu'\kappa}(y)$$

$J_{\mu\nu\kappa}$ is some expression linear in $H$ and $G^{ab}_{\mu\nu,\mu'\nu'}(x, y)$ is the Green function satisfying

$$\partial_5 G^{ab\mu'\nu'}_{\mu\nu} - \frac{1}{2} \epsilon_{\mu\nu}^{\alpha\beta\gamma} (D^{(y)}_\alpha)_{\beta\gamma}^a G^{cb\mu'\nu'}_{\beta\gamma} = \delta^{\mu'\nu'}_{\mu\nu} \delta^{ab} \delta^{(6)}(x - y)$$

with the BC: $G^{ab\mu'\nu'}_{\mu\nu}(x, y) = 0, \quad |x_5| \to \infty$.

- As before, the commutator closes to the standard 5d Lorentz transformation plus terms vanishing onshell, plus a gauge transformation.
Note that our proposed Lorentz symmetry is nonlocal.

This is not unexpected since we are working in a gauge fixed formulation without $B_{\mu 5}$. The Lorentz symmetry is also nonlocal for QED in Coulomb gauge or string in lightcone gauge.
Property 4: Reduction to D4-branes

- Consider a compactification of $x_5$ on a circle of radius $R$. The dimensional reduced action reads
  \[ S = \frac{2\pi R}{2} \int d^5x \, \text{tr} \left( -\tilde{H}_{\mu\nu}^2 + (F_{\mu\nu} - 2\pi Rc\tilde{H}_{\mu\nu})E_{\mu\nu} \right) \]

- Integrate out $E_{\mu\nu}$, we obtain
  \[ F_{\mu\nu} = 2\pi Rc\tilde{H}_{\mu\nu}. \]

  and eliminate $\tilde{H}_{\mu\nu}$, we obtain the 5d Yang-Mills action
  \[ S_{YM} = -\frac{1}{4\pi Rc^2} \int d^5x \, \text{tr} \, F_{\mu\nu}^2. \]

- This gives the YM coupling and the gauge group to be
  \[ g_{YM}^2 = Rc^2, \quad G = U(N) \]

  for a system of $N$ M5-branes.

However there is a subtlety.
Property 4: Reduction to D4-branes

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However there is a subtlety.
EOM gives \( D^\mu F_{\mu\nu} = 0 \) instead of

\[
D_\mu F^{\mu\nu} = -\frac{\pi R}{2} \epsilon^{\nu\alpha\beta\gamma\delta} [F_{\alpha\beta}, B_{\gamma\delta}]?
\]

Need to be more careful with the implementation of Delta function:

\[
\int [DA][DB][DE] e^{-S} = \int [DA][DB] e^{-S_{YM}} \delta(F_{\mu\nu} - 2\pi R \tilde{H}_{\mu\nu}) = \int [DA] e^{-S_{YM} - S'},
\]

where consistency requires that

\[
\frac{\delta S'}{\delta A_\nu} = \frac{1}{2} \epsilon^{\nu\alpha\beta\gamma\delta} [F_{\alpha\beta}, B_{\gamma\delta}]
\]

The 5d theory is thus given by the action \( S_{5d} = S_{YM} + S' \).

\( S' \) describes a higher derivative correction term to the Yang-Mills theory since \([F, B] \sim DDB\) and \( B \) is of the order of \( F \).

It might be possible that \( S' \) captures the non-abelian DBI action of D4-branes.
We have constructed a non-abelian action of tensor fields with the properties:

1. the action admits a self-duality equation of motion,
2. the action has manifest 5d Lorentz symmetry and a modified 6d Lorentz symmetry,
3. on dimensional reduction, the action gives the 5d Yang-Mills action plus corrections.

Based on these properties, we propose our action to be the bosonic theory describing the gauge sector of coincident M5-branes in flat space.

We still need to include the scalar field and fermions, and to susy complete the action.

Another interesting direction is to explore the dynamical content of the theory.
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M2-branes can end on M5-brane. The endpoint gives strings living on the M5-brane. These self-dual strings appear as solitons of the M5-brane theory.

In the Abelian case, the self-dual string solition has been obtained by Perry-Schwarz (1996) and also by Howe-Lambert-West (1997).
Self-Dual String Soliton in the Perry-Schwarz Theory

- The Perry-Schwarz non-linear field equation is given by
\[
\tilde{H}_{\mu\nu} = \frac{(1 - y_1) H_{\mu\nu 5} + H_{\mu\rho 5} H^{\rho\sigma 5} H_{\sigma\nu 5}}{\sqrt{1 - y_1 + \frac{1}{2} y_1^2 - y_2}},
\]
where \( y_1 := -\frac{1}{2} H_{\mu\nu 5} H^{\mu\nu 5} \), \( y_2 := \frac{1}{4} H_{\mu\nu 5} H^{\nu\rho 5} H_{\rho\sigma 5} H^{\sigma\mu 5} \).

- A solution aligning in the \( x^5 \) direction is solved by the ansatz:
\[
B = \alpha(\rho) dt dx^5 + \frac{\beta}{8} (\pm 1 - \cos \tilde{\theta}) d\tilde{\phi} d\tilde{\psi},
\]
where the 6d metric is
\[
ds^2 = -dt^2 + (dx^5)^2 + d\rho^2 + \rho^2 d\Omega_3^2,
\]
with the three-sphere given in Euler coordinates
\[
d\Omega_3^2 = \frac{1}{4} [(d\tilde{\psi} + \cos \tilde{\theta} d\tilde{\phi})^2 + (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2)],
\]
- Note that \( B_{\tilde{\phi}\tilde{\psi}} = \beta(\pm 1 - \cos \tilde{\theta}) \) is the potential of a Dirac monopole!
For this ansatz, \( y_1 = \alpha'^2 \), \( y_2 = \alpha'^4/2 \) and the EOM reads

\[
\alpha'(\rho) = \frac{\beta}{\sqrt{\beta^2 + \rho^6}}.
\]

The solution is regular everywhere:

\[
\alpha \sim \rho \quad \text{as} \quad \rho \to 0,
\]

\[
\alpha \sim -\frac{\beta}{2\rho^2} + \text{const.} \quad \text{as} \quad \rho \to \infty.
\]

The magnetic charge \( P \) and electric charge \( Q \) (per unit length) of the string:

\[
P = \int_{S^3} H, \quad Q = \int_{S^3} *H,
\]

gives

\[
P = Q = 2\pi^2 \beta
\]

hence the string is self-dual.
charge quantization condition for self-dual string in 6d

\[ PQ' + QP' \in 2\pi \mathbb{Z} \]

(Deser, Gomberoff, Henneaux, Teitelboim)

for the self-dual string gives

\[ \beta = \pm \sqrt{\frac{n}{4\pi^3}}, \]

i.e.

\[ P = Q = \pm \sqrt{n\pi}. \]
For the non-abelian theory, the equations of motion to be solved are:

\[ \tilde{H}_{\mu \nu} = \partial_5 B_{\mu \nu} \]
\[ F_{\mu \nu} = c \int dx_5 \tilde{H}_{\mu \nu} \]

Need to do:

1. construct a string aligns in a different direction, say \( x^4 \).
   The gauge field is auxiliary:
   \[ F_{\mu \nu} = c \int dx_5 \tilde{H}_{\mu \nu} = c(B_{\mu \nu}(x_5 = \infty) - B_{\mu \nu}(x_5 = -\infty)) \]
   Therefore if the non-Abelian solution is translationally invariant along \( x^5 \), then \( F_{\mu \nu} = 0 \) is trivial.

2. supersymmetrize: Perry-Schwarz self-dual string solution is non-BPS as there is no other matter field turned on to cancel the tensor field force.
Self-dual string in $x^4$ direction

It is instructive to first obtain the PS solution in Cartesian coordinates for a string in the $x^4$ direction. The result is

$$H = \frac{\alpha'}{\rho} \, dt dw (xdx + ydy + zdz + x^5 dx^5) + \frac{\beta}{\rho^4} (x^5 dxdyz - xdydzx^5 + zdydx^5 - ydzdx^5),$$
To get $B$, we integrate directly the self-duality equation

$$H_{\mu \nu 5} = \partial_5 B_{\mu \nu}$$

and get

$$B_{ij} = -\frac{1}{2} \frac{\beta \epsilon_{ijk} x_k}{r^3} \left( \frac{x^5 r}{\rho^2} + \tan^{-1}(x^5/r) \right), \quad B_{tw} = -\frac{\beta}{2 \rho^2},$$

$i, j = 1, 2, 3, w = x^4$.

Although the auxiliary field does not appear in the PS construction, it is amazing that

$$F_{ij} = -\frac{c \beta \pi}{2} \frac{\epsilon_{ijk} x_k}{r^3}, \quad F_{tw} = 0$$

i.e. a Dirac monopole in the $(x, y, z)$ subspace if $c \beta = -2/\pi$!

The presence of a Dirac monopole was already apparent in the original solution of PS. Here, we reveal that the monopole configuration also appears (more directly) in the auxiliary gauge field.
It turns out the use of an non-abelian monopole in place of the Dirac monopole is precisely what is needed to construct the non-abelian self-dual string solution.

Here we have two candidates of non-abelian monopole: the Wu-Yang monopole and the ’t Hooft-Polyakov monopole.
Non-abelian Wu-Yang & ’t Hooft-Polyakov monopole

Wu-Yang

- Consider $SU(2)$ gauge group

$$[T^a, T^b] = i \epsilon^{abc} T^c, \quad a, b, c = 1, 2, 3.$$  

- The non-abelian Wu-Yang monopole is given by

$$A_i^a = -\epsilon_{aik} \frac{x_k}{r^2}, \quad F_{ij}^a = \epsilon_{ijm} \frac{x_m x_a}{r^4}, \quad i, j = 1, 2, 3.$$  

- Note that the field strength for the Wu-Yang solution is related to the field strength of the Dirac monopole by a simple relation:

$$F_{ij}^a = F_{ij}^{(Dirac)} \frac{x^a}{r}.$$  

- Note that the color magnetic charge vanishes

$$\int_{S^2} F^a = 0$$

and the Wu-Yang solution is actually not a monopole. Nevertheless it plays a key role in the construction of non-abelian monopole of ’t Hooft-Polyakov.
In the BPS limit, the 't Hooft-Polyakov monopole satisfies

\[ \frac{1}{2} \epsilon_{ijk} F_{ij} = D_k \phi, \quad D_k^2 \phi = 0 \]

where \( \phi \) is an adjoint Higgs scalar field.

The solution is given by

\[ A^a_i = -\epsilon_{aik} \frac{x^k}{r^2} (1 - k_v(r)), \quad \phi^a = \frac{v x^a}{r} h_v(r), \]

where

\[ k_v(r) := \frac{vr}{\sinh(vr)}, \quad h_v(r) := \coth(vr) - \frac{1}{vr}. \]

Asymptotically \( r \to \infty \), we have

\[ A^a_i \to -\epsilon_{aik} \frac{x^k}{r^2}, \quad \phi^a \to \frac{|v| x^a}{r} := \phi_\infty, \]

i.e. precisely the Wu-Yang monopole at infinity.
Unbroken $U(1)$ gauge symmetry at infinity may be identified as the electromagnetic gauge group and the electromagnetic field strength can be obtained as a projection:

$$\mathcal{F}_{ij} = F_{ij}^a \phi^a = \epsilon^{ijk} \frac{x^k}{r^3}, \quad \text{for large } r.$$ 

The magnetic charge is given by $p = \int_{S^2} \mathcal{F} = 4\pi$, which corresponds to a magnetic monopole of unit charge.

Note that at the core $r \to 0$, we have

$$A_i \to 0, \quad \phi \to 0$$

and hence the $SU(2)$ symmetry is restored at the monopole core.
Non-abelian self-dual string solution

Inspired by the relation of Dirac monopole to the Wu-Yang solution, try the ansatz

\[ H_{\mu \nu \lambda}^a = H_{\mu \nu \lambda}^{(PS)} \frac{X^a}{r} \]

Here \( r = \sqrt{x^2 + y^2 + z^2} \) and \( H_{\mu \nu \lambda}^{(PS)} \) is the field strength for the linearized Perry-Schwarz solution aligning in the \( x^4 \) direction. Self-duality is automatically satisfied!

\[ B_{\mu \nu} = B_{\mu \nu}^{(PS)} \frac{X^a}{r}, \]

It is amusing that the auxiliary field configuration is given by

\[ F_{ij}^a = -\frac{c \beta \pi}{2} \frac{\epsilon_{ijm}X_m X_a}{r^4}, \quad F_{tw}^a = 0. \]

This is the Wu-Yang monopole if we take \( c \beta = -\frac{2}{\pi} \).
Like the Wu-Yang monopole, the color magnetic charge of our Wu-Yang string solution vanishes.

However this is not a problem as we should include scalar fields (which are natural from the point of view of (2,0) supersymmetry).

Although we do not have the full (2,0) supersymmetric theory, one can argue (a simple dimensional analysis) that the self-duality equation of motion is not modified by the presence of the scalar fields.

As for the scalar field’s EOM, the self-interacting potential vanishes if there is only one scalar field turned on (R-symmetry). As a result, the equation of motion of the scalar field is

$$D_M^2 \phi = 0.$$
A reasonable form of the BPS equation is the non-abelian generalization of the BPS equation of Howe-Lambert-West:

\[ H_{ijk} = \epsilon_{ijk} \partial_5 \phi, \quad H_{ij5} = -\epsilon_{ijk} D_k \phi. \]

This follows immediately from the supersymmetry transformation

\[ \delta \psi = (\Gamma^M \Gamma^I D_M \phi^I + \frac{1}{3!2} \Gamma^{MNP} H_{MNP}) \epsilon \]

and the 1/2 BPS condition

\[ \Gamma^{046} \epsilon = -\epsilon. \]

Note: this is the most natural non-abelian generalization of the abelian (2,0) supersymmetry transformation.
The BPS equation can be solved with
\[ \phi^a = -(u + \frac{\beta}{2\rho^2}) \frac{x^a}{r}, \]

The transverse distance \(|\phi|\) defined by \(|\phi|^2 = \phi^a \phi^a\) gives
\[ |\phi| = |u + \frac{\beta}{2\rho^2}|. \]

This describes a system of M5-branes with a spike at \(\rho = 0\) and level off to \(u\) as \(\rho \to \infty\). Hence the physical interpretation of our self-dual string is that two M5-branes are separating by a distance \(u\) and with an M2-brane ending on them.
Asymptotic $U(1)$ $B$-field is $B_{\mu\nu} \equiv \hat{\phi}_\infty^a B_{\mu\nu}^a = \pm B_{\mu\nu}^{(PS)}$ and we obtain

$$P = Q = -\frac{4\pi}{|c|}.$$ 

Charge quantization yields

$$c = \pm 4\sqrt{\pi}$$
One may also consider the compactified case with $x^5$ compactified on a circle.

Fourier mode expansion

$$H_{MNP} = \sum_n e^{inx^5/R} H^{(n)}_{MNP}(r).$$

Integrating $H_{\mu\nu 5} = \partial_5 B_{\mu\nu}$, we get

$$B_{\mu\nu} = \frac{x^5}{2\pi Rc} F_{\mu\nu}(r) + \sum_{n=-\infty}^{\infty} e^{inx^5/R} B^{(n)}_{\mu\nu}(r),$$

where

$$H^{(n\neq 0)}_{\mu\nu 5}(r) = \frac{in}{R} B^{(n\neq 0)}_{\mu\nu}(r).$$

Note the presence of winding modes.
Consider an ansatz with the only nonzero components $B_{tw}$ and $B_{ij}$, the self-duality condition reads

$$D_{[i} B_{jk]}^{(0)} = \epsilon_{ijk} \frac{F_{tw}}{2\pi Rc}, \quad D_k B_{tw}^{(0)} = -\frac{f_k}{2\pi Rc}$$

$$D_k b_k^{(n)} = \frac{in}{R} B_{tw}^{(n)}, \quad D_k B_{tw}^{(n)} = -b_k^{(n)} \frac{in}{R}, \quad n \neq 0,$$

where

$$f_k(r) := \frac{1}{2} \epsilon_{ijk} F_{ij} \quad \text{and} \quad b_k^{(n)}(r) := \frac{1}{2} \epsilon_{ijk} B_{ij}^{(n)} \quad \text{for} \ n \neq 0.$$

Note the red equation takes exactly the same form as the BPS equation for the HP monopole if we identify $-2\pi Rc B_{tw}^{(0)} := \phi$ as the scalar field there.

Hence it can be solved with $A_i^a$ taken to be that of the HP monopole

$$A_i^a = -\epsilon_{aik} \frac{x^k}{r^2} (1 - k_v(r)).$$

This implies $F_{tw} = 0$. 
The whole system of equations can be solved exactly and we obtain

\[
B_{tw}^a = -\frac{h_v(r) \nu x^a}{2\pi Rc} \frac{\nu x^a}{r} + \sum_{n \neq 0} \alpha_n e^{i n x^5 / R} \frac{e^{-|n|r/R}}{vr} \left(1 + \frac{vR}{|n|} \coth(vr)\right) \frac{\nu x^a}{r},
\]

\[
B_{ij}^a = \frac{x^5}{2\pi Rc} F_{ij}^a(r) + c_0 F_{ij}^a(r) + \sum_{n \neq 0} e^{i n x^5 / R} B_{ij}^a(n)(r),
\]

where

\[
B_{ij}^a(n) = \epsilon_{ijk} \frac{R}{in} \left(-\nu^3 (r a'_n - k_v(r) a_n) \frac{x^k x^a}{r} - \delta^a_k a_n k_v(r) \frac{\nu}{r}\right), \quad n \neq 0.
\]

As before, we can include a scalar field \( \phi \) as needed in the (2,0) theory. The charge is found to be the same as the uncompactified string (as it should be by consistency).
Outline

1. Introduction

2. Non-abelian action for multiple M5-branes
   - Perry-Schwarz action for a single M5-brane

3. Non-abelian self-dual string solution

4. Discussions
I. We have constructed a non-abelian action of tensor fields that we propose to be the bosonic theory describing the gauge sector of coincident M5-branes in flat space.

II. We have also constructed the non-abelian string solutions of the theory. By including a scalar field, we argue how the solution can be promoted to become a solution of the (2,0) theory. This provides dynamical support to our proposed theory.
Further questions

- Supersymmetry: (2,0)? (1,0)?
  Scalar potential and BPS equation?
- Other solutions? BPS string junction?
- Integrability?
- Covariant PST extension of our model?
- Connection with the 5d SYM proposal of Douglas and Lambert-Papageorgakis-Schmidt-Summerfield?
  Where is the $B$-field in the SYM description? (similar to the problem of extracting the gravity field in the BFSS matrix model?)